LEFSCHETZ PROPERTIES IN ALGEBRA, GEOMETRY, TOPOLOGY AND COMBINATORICS
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Lefschetz properties in algebra, geometry and combinatorics
notes for the preparatory school

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1. Introduction

These notes represent background and supplementary material for our course in the Preparatory School for the conference "Lefschetz Properties in Algebra, Geometry and Combinatorics," which will be held in Kraków in June, 2024.

The Lefschetz properties represent some "expected" behavior of the multiplication on a graded module over a homogeneous polynomial ring. It is an extremely natural and basic idea, and as a result it shows up in many fields, in different guises, and it has many consequences (as is suggested by the title of the conference). But it is also true that in many situations the actual behavior is different from the expected one. As a result, we will see situations where we want to prove that the properties hold, but also many in which we will have to show that the expectations do not materialize. There are three courses in this school, and in them you will see different manifestations of these basic principles.

A fundamental piece of information about a graded module that is behind the Lefschetz properties is its Hilbert function. This is essentially a way of measure "how big" the components of the graded module are. It will play an important role in this course, and probably in all three courses. The Hilbert function is an important invariant for a projective variety or, more precisely, of a standard graded algebra (or, even more generally, of a finitely generated graded module). It is an interesting question to determine what the known properties of the variety or algebra imply about the Hilbert function, and perhaps an even more interesting question to determine the converse, i.e. what information the Hilbert function can give about the geometry associated to the variety or algebra.

We have tried to include a fairly large number of exercises. The problem-solving sessions will focus on the assigned exercises. The solutions are at the end of the notes (after the references), but we strongly encourage you to spend a lot of time working them out before looking at the solutions.

We assume familiarity with subvarieties of affine and projective space. The book [CLO] gives a lot of the necessary background, and we include some exercises involving that material.

We will assume that you have some familiarity with the following topics:

- affine and projective spaces
- homogeneous coordinates for projective space
- the projectivization of a vector space
- duality for projective spaces
- affine and projective varieties, hypersurfaces
- monomial ideals
- minimal free resolutions

although we will review some of these notions in these notes. Some of the results that we'll talk about depend on the field k that we are using. Unless stated otherwise. assume that k is algebraically closed and of characteristic zero.

In these notes we will sometimes need to mention and use some facts, even if we avoid their proofs. For the most part, these facts are placed into "Remarks." The converse is not true, though: not all Remarks in these notes mean that their content is a fact that we will not prove. Sometimes a remark is just a remark.

There are three papers attached to these notes. The first is a joint paper by Migliore and Uwe Nagel [MN1], which is an expository overview of the Lefschetz properties as they appear in various fields. The second is a joint paper of Migliore with Tadahito Harima, Uwe Nagel and Junzo Watanabe [HMNW]. This was one of the first papers to deal with the Lefschetz properties, and in particular this paper introduced the use of the syzygy bundle to prove the WLP for codimension 3 complete intersections. In addition, it characterized the Hilbert functions of algebras with WLP or SLP (same answer!), and described bounds on the Betti numbers for algebras with WLP. The third paper, [JM], is a nice overview of some of the open problems in the theory of Lefschetz Properties, written by Martina Juhnke-Kubitzke and Rosa María Miró-Roig.

2. Background and exercises

This section contains some exercises and remarks to help make sure you have the needed background. The solutions can be found starting on page 53. Two useful references for this material are [CLO] and [AM].

2.1. Basics on varieties and rings.

Exercise 1. Prove that in the ring $R = k[x_1, \ldots, x_n]$ there are $\binom{d+n-1}{n-1}$ monomials of degree d for any $d \geq 1$ and $n \geq 1$.

Exercise 2. Let $R = k[x, y]$ where k is an infinite field of characteristic $\neq 2$. Prove:

(a) $\langle x + y, x - y \rangle = \langle x, y \rangle$.
(b) $\langle x, y \rangle = \langle x + xy, y + xy, x^2, y^2 \rangle = \langle x + xy, y + xy, x^2 \rangle$.
(c) In the last equality of (b), show that the three generators are irredundant (i.e. if you remove any one of them, the ideal becomes smaller).

Exercise 3. Let $V = \mathbb{V}(f_1, \ldots, f_s)$ and $W = \mathbb{V}(g_1, \ldots, g_t)$ be varieties in the affine space k^n. Prove that

$$V \cap W = \mathbb{V}(f_1, \ldots, f_s, g_1, \ldots, g_t).$$

Exercise 4. Prove that any finite union of points in $\mathbb{A}^n$ is an affine variety.

Exercise 5. Let $k = \mathbb{R}$. Let Z be the set of all points in $\mathbb{R}^2$ with integer coordinates.

(a) Let $f(x, y)$ be a polynomial vanishing at every point of Z. Prove that $f(x, y)$ must be the zero polynomial. [Hint: if $f(x, y)$ vanishes at every point of Z, what can you say about $f(x, 0)$?]

(b) Conclude that Z is not an affine variety.

Exercise 6. Prove that

$$X = \{(m, m^3 + 1) \in \mathbb{R}^2 \mid m \in \mathbb{Z}\}$$

is not an affine variety.

Exercise 7. Let k be a field and let V be a subset of k^1. Prove the following statement:

V is a subvariety of k^1 if and only if V is a finite set of points in k^1.

Note that you have to prove both directions.

Exercise 8. Let $\mathbb{F}_p$ be the field with p elements, for any prime p.

(a) Consider the polynomial $g(x, y) = x^2y + y^2x \in \mathbb{F}_2[x, y]$. Prove that $g(a, b) = 0$ for all $(a, b) \in \mathbb{F}_2^2$.

(b) Find a nonzero polynomial $g(x_1, \ldots, x_n) \in \mathbb{F}_2[x_1, \ldots, x_n]$ involving all n variables, such that $g(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in \mathbb{F}_2^n$.

(c) Repeat (a) and (b), replacing $\mathbb{F}_2$ by $\mathbb{F}_p$.

Exercise 9.

(a) Let S be a set in k^n. (If it helps, just think about $\mathbb{R}^2$.) Show that

$$S \subseteq \mathbb{V}(\mathbb{I}(S)).$$

(b) Give an example to show that the inclusion in part (a) is not necessarily an equality. If you want, you can use the following example, as long as you completely justify why it answers the question!

$$S_1 = \cup\{(0,i) \mid i \in \mathbb{Z}\} = \{\dots,(0,-2),(0,-1),(0,0),(0,1),(0,2),\dots\} \subset \mathbb{R}^2.$$

You'll have to explicitly compute $\mathbb{I}(S_1)$, and then $\mathbb{V}(\mathbb{I}(S_1))$.

(c) However, if S happens to be a *variety* then show that it is true that

$$S = \mathbb{V}(\mathbb{I}(S)).$$

Exercise 10. Show that if V is any affine variety in k^n then $\mathbb{I}(V)$ is a *radical* ideal. This means that if $f^m \in \mathbb{I}(V)$ for some m then $f \in \mathbb{I}(V)$.

Exercise 11. Let I and J be ideals in $k[x_1,\dots,x_n]$. We define

$$I \cap J = \{f \in R \mid f \in I \text{ and } f \in J\}.$$

We define IJ to be the set of polynomials that can be written as finite sums in the following way:

$$IJ = \left\{ \sum_{i=1}^{m} f_i g_i \mid f_i \in I, g_i \in J \right\}.$$

(a) Prove that $I \cap J$ is an ideal.

(b) Prove that IJ is an ideal.

(c) Show that $IJ \subseteq I \cap J$ (as ideals).

(d) Give an example to show that IJ is not necessarily equal to $I \cap J$. Justify your answer!

(e) If I and J are ideals in $k[x_1,\dots,x_n]$, prove that $\mathbb{V}(IJ) = \mathbb{V}(I) \cup \mathbb{V}(J)$. [Hint: this is closely related to our proof that $\mathbb{V}(I) \cup \mathbb{V}(J)$ is again an affine variety.]

(f) If I and J are ideals in $k[x_1,\dots,x_n]$, prove that $\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J)$. Combined with the previous part, conclude that $\mathbb{V}(IJ) = \mathbb{V}(I \cap J)$.

Exercise 12. Let $\phi = [F_1,\dots,F_m] : \mathbb{C}^n \to \mathbb{C}^m$, where $F_1,\dots,F_m \in \mathbb{C}[x_1,\dots,x_n]$. Let $X = \mathbb{V}(G_1,\dots,G_k)$ be a subvariety of $\mathbb{C}^m$ (so $G_1,\dots,G_k \in \mathbb{C}[y_1,\dots,y_m]$). Prove that

$$\phi^{-1}(X) = \mathbb{V}(G_1(F_1,\dots,F_m),\dots,G_k(F_1,\dots,F_m)).$$

(Make sure you prove both inclusions.)

2.2. **Noetherian rings.** A useful source for Noetherian rings is [AM].

Exercise 13. Prove that $k[x_1,\dots,x_{n-1}][x_n] \cong k[x_1,\dots,x_n]$.

Definition 2.1. A ring A is *Noetherian* if it satisfies any of the following equivalent conditions.

(a) Every non-empty set of ideals in A has a maximal element with respect to inclusion.
(b) Every ascending chain of ideals in A stabilizes.
(c) Every ideal in A is finitely generated.

Condition (b) above is called the *Ascending Chain Condition (ACC)*.

Remark 2.2. The equivalence of (a), (b) and (c) is proved, for example, in [AM] Chapter 6. The following statements are also true.

1. If A is Noetherian then so is the polynomial ring $A[x]$. (This is the famous Hilbert Basis Theorem.) Using Exercise 13, this implies that

   the polynomial ring $R = k[x_0, \dots, x_n]$ *is Noetherian*

   ([AM] Theorem 7.5).
2. If A is Noetherian and $\phi : A \to B$ is an epimorphism then B is Noetherian ([AM] Proposition 7.1). This implies that

   any quotient R/I *is also Noetherian.*

It follows from all this that whenever we have an ideal I in a polynomial ring $R = k[x_1, \dots, x_n]$ over a field k then I is finitely generated. This is very useful!

Exercise 14. Consider the set of polynomials $f_i \in R = k[w, x, y, z]$ defined by

$$f_i = w^i + x^{i+1} + y^{i+2} + z^{i+7}$$

for all $i \geq 1$. Prove that there exists an integer N such that for $i \geq N$, f_i is a linear combination (with coefficents in R) of $f_1, \dots, f_{N-1}$. (We do not want to know a precise value of N.)

2.3. More background from [CLO] on affine varieties and ideals.

Exercise 15. In this problem we will work over the field of real numbers, $\mathbb{R}$.

(a) Let $I = \langle f_1, \dots, f_s \rangle$ be any ideal in $\mathbb{R}[x_1, \dots, x_n]$. Let $V = \mathbb{V}(I) \subset \mathbb{R}^n$ be the corresponding variety. Find a single polynomial f such that $V = \mathbb{V}(f)$. Prove your answer.

(b) Let $I = \langle f_1, \dots, f_s \rangle$ be any ideal in $\mathbb{R}[x_1, \dots, x_n]$. Suppose that $\mathbb{V}(I) = \emptyset$. Show that there is at least one element of I that has no zero in $\mathbb{R}^n$. Justify your answer. (Notice that $\mathbb{R}$ is not algebraically closed, so you can't use the Nullstellensatz.)

Exercise 16. Let V and W be varieties in $\mathbb{C}^n$ such that $V \cap W = \emptyset$. Prove that there exist $f \in \mathbb{I}(V)$ and $g \in \mathbb{I}(W)$ such that $f + g = 1$.

Exercise 17. Let $I \subset k[x_1, \dots, x_n]$ be an ideal. Let $\sqrt{I}$ be its radical. Show that there is a positive integer p such that for every $f \in \sqrt{I}$, $f^p \in I$. (The thing to stress is that the choice of p does not depend on what f you choose; rather, p depends only on what $\sqrt{I}$ is.) [Hint: $\sqrt{I}$ is an ideal in a Noetherian ring.]

Exercise 18. Let I and J be ideals in $\mathbb{C}[x_1, \dots, x_n]$ such that

$$I + J = \langle 1 \rangle = \mathbb{C}[x_1, \dots, x_n].$$

(a) Prove that the varieties $\mathbb{V}(I)$ and $\mathbb{V}(J)$ are disjoint.

(b) Prove that $IJ = I \cap J$.

(c) Part (b) depends very much on the assumption $I + J = \langle 1 \rangle$. Give an example of ideals I and J not satisfying that property, for which it is *not* true that $IJ = I \cap J$.

Exercise 19. For each of the following, R is the polynomial ring $k[x_1, \dots, x_n]$ and X is an algebraic set in $\mathbb{A}^n_k$, where k is a field. Any extra assumptions about k will be given explicitly. For each part, give the indicated example **or show that no such example exists**. When you give an example, you are allowed to choose a specific field k and a specific value of n if you want to (e.g. taking $k = \mathbb{R}$ and $n = 2$ may be easier to visualize).

[Hint: for two of these the answer is "no" (so you have to prove that J doesn't exist), and the rest are "yes" (so you have to find such an example). **All** of these should be very short answers!!]

(a) Does there exist an ideal $J \subset R$ such that $J = \mathbb{I}(X)$ for some algebraic set X, but J is not radical?

(b) Does there exist an ideal $J \subset R$ such that $J = \mathbb{I}(X)$ for some algebraic set X, but J is not prime?

(c) Does there exist a **prime** ideal $J \subset R$ which is not maximal?

(d) Does there exist an ideal J that is not prime, but $\mathbb{I}(\mathbb{V}(J))$ is prime?

(e) Does there exist an ideal J and a polynomial $f \in R$ such that f vanishes at every point of $\mathbb{V}(J)$, but $f \notin J$?

(f) Assume that k is algebraically closed. Does there exist an ideal J and a polynomial $f \in R$ such that f vanishes at every point of $\mathbb{V}(J)$, but *no power* of f is in J?

Exercise 20. Let I, J be ideals in $k[x_1, \dots, x_n]$ and suppose that $I \subset \sqrt{J}$. Show that $I^m \subset J$ for some integer $m > 0$.

2.4. **Background from [CLO] on projective varieties and homogeneous ideals.** For convenience in this section our polynomial ring will be $R = k[x_0, \dots, x_n]$ (i.e. we start with x_0 instead of x_1), so that we can talk about varieties in $\mathbb{P}^n$.

Definition 2.3. Given a monomial $x_0^{m_0} \dots x_n^{m_n}$, its *degree* is $m_0 + \dots + m_n$. Any polynomial can be written as a linear combination $a_0 M_0 + \dots + a_N M_N$ of distinct monomials in a unique way. For any i, $a_i x_i^{m_i}$ is called a *term*. A polynomial is said to be *homogeneous* if all the terms have the same degree. Any polynomial f can be written as the sum of homogeneous polynomials: $f = f_0 + f_1 + \dots + f_d$ in a unique way; the f_i are called the *homogeneous components* of f. A homogeneous polynomial f of degree d is also called a *form* of degree d.

Definition 2.4. An ideal $I \subset k[x_0, \dots, x_n]$ is *homogeneous* if, for each $f \in I$, the homogeneous components of f are also in I.

Theorem 2.5. *Let $I \subset k[x_0, \dots, x_n]$ be an ideal. The following are equivalent:*
(i) I is a homogeneous ideal.
(ii) There exists a set of homogeneous polynomials $f_1, \dots, f_s$ that generate I.

Exercise 21. Let I and J be homogeneous ideals in $k[x_0, \dots, x_n]$.

(a) Prove that $I + J$ is homogeneous.

(b) Prove that $I \cap J$ is homogeneous.

Exercise 22. Homogeneous polynomials satisfy an important relation known as *Euler's Theorem*. It says the following. For convenience assume that our field is $\mathbb{R}$. Let $f \in \mathbb{R}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d. Then

$$\sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$

(a) Illustrate Euler's theorem by cooking up a homogeneous polynomial, f, having three terms and showing that the theorem is true for your example.

(b) Prove Euler's theorem by considering $f(\lambda x_0, \ldots, \lambda x_n)$ as a function of λ, and differentiating with respect to λ using the chain rule.

(c) Let $R = \mathbb{R}[x, y, z]$ and let $f = xyz$. In $\mathbb{P}^2_{\mathbb{R}}$ describe $\mathbb{V}(f)$, $\mathbb{V}(f_x, f_y, f_z)$, and the relation between these two varieties. (Here f_x, f_y, f_z are the partials with respect to x, y, z respectively.) How is Euler's theorem relevant to this last part?

(d) Let $R = \mathbb{R}[x, y, z]$ and let $f = xyz(x + y + z)$. In $\mathbb{P}^2_{\mathbb{R}}$ describe $\mathbb{V}(f)$, $\mathbb{V}(f_x, f_y, f_z)$, and the relation between the two. Again, how is Euler's theorem relevant to this last part?

Exercise 23. Recall that for an ideal $I \subset k[x_0, \ldots, x_n]$, a set of polynomials $f_1, \ldots, f_r$ are *minimal generators* for I if $I = \langle f_1, \ldots, f_r \rangle$, and if the removal of any of the f_i changes the ideal. We also say that $\{f_1, \ldots, f_r\}$ form a *minimal generating set* for I. For example, for $I = \langle x^2, y^2, (x+y)(x-y) \rangle \subset k[x, y, z]$, the generators are not minimal since $(x+y)(x-y) = x^2 - y^2$, so removing $(x+y)(x-y)$ does not change the ideal.

(a) Give an example of an ideal $I \subset \mathbb{C}[x, y, z]$ such that
- I has a minimal generating set consisting of five homogeneous polynomials;
- $\mathbb{V}(I) = \emptyset$;
- The five generators of I all have different degrees.

(Hint: think about monomial ideals.)

(b) In the statement of the Projective Weak Nullstellensatz ([CLO] Chapter 8, Section 3, Theorem 8), the authors mention integers m_i (in part (iii)) and r (in part (iv)). For your answer to part (a), what are the values of m_1, m_2, m_3 and r? Be sure to justify your answer.

(c) Find a counterexample to the following statement:

If I is a homogeneous ideal and J is an ideal such that $J \subset I$ then J is homogeneous.

Exercise 24. Let ϕ be an automorphism of $\mathbb{P}^2$. What this means is that there is some *invertible* 3×3 matrix A such that for $P = [p_1, p_2, p_3]$,

$$\phi(P) = A \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

Let P, Q, R be three points in $\mathbb{P}^2$. Show that if P, Q, R are collinear then $\phi(P), \phi(Q), \phi(R)$ are collinear. Is the converse true?

Exercise 25. In this problem, we will be talking about planes, Λ, in $\mathbb{P}^n$. You can assume that the field in question is $\mathbb{R}$, the real numbers. Remember that, in $\mathbb{P}^2$, the only possibility for Λ is that it is all of $\mathbb{P}^2$. In $\mathbb{P}^3$, Λ is the vanishing locus of a single homogeneous linear polynomial L, and we have $I_\Lambda = \langle L \rangle$. You can freely use these facts.

(a) Describe the homogeneous ideal of a plane in $\mathbb{P}^4$ in terms of the minimal generators of its ideal (no proof required).

(b) Let Λ_1 and Λ_2 be distinct planes in $\mathbb{P}^3$. Prove that $\Lambda_1 \cap \Lambda_2$ *must* be a line.

(c) Give an example of two distinct planes, Λ_1 and Λ_2, in $\mathbb{P}^4$ whose intersection is the point $[1, 1, 1, 1, 1]$.

(d) In part c), is your answer unique, or are there finitely many possible answers, or are there infinitely many possible answers? Explain.

Exercise 26. A beautiful fact about projective space is the notion of **duality**. Let's limit ourselves to $\mathbb{P}^2_\mathbb{R}$, the real projective plane. (We will understand that we are working over $\mathbb{R}$ and not bother writing the subscript $\mathbb{R}$ each time.)

Recall that a line ℓ in $\mathbb{P}^2$ is the vanishing locus of a homogeneous linear polynomial, i.e. $\ell = \mathbb{V}(ax + by + cz)$ for some choice of $a, b, c \in \mathbb{R}$ not all zero.

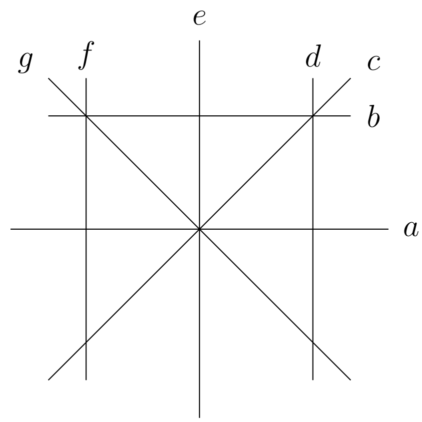
(a) Show that $ax + by + cz = 0$ defines the same line as $3x + 4y + 5z = 0$ if and only if there exists some $t \in \mathbb{R}$ such that $a = 3t$, $b = 4t$ and $c = 5t$. (Of course 3, 4, 5 is just an example.) [Hint: $\Leftarrow$ is almost immediate. For $\Rightarrow$, you can use the fact that in $\mathbb{P}^2$, either two lines meet at a single point or they are the same line. It may help to take the linear algebra point of view.]

(b) Based on (a), show that the **set** of lines in $\mathbb{P}^2$ itself can be viewed as a projective plane, which we will denote by $(\mathbb{P}^2)^\vee$.

(c) Let P_1, P_2, P_3 be points of $(\mathbb{P}^2)^\vee$ and let $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ be the lines in $\mathbb{P}^2$ that they correspond to. Show that P_1, P_2, P_3 all lie on a line in $(\mathbb{P}^2)^\vee$ if and only if $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ all pass through a common point. [Hint: if you look at the equation $ax+by+cz = 0$, you can think of a, b, c as given and x, y, z as the variables, OR you can think of x, y, z as given and a, b, c as the variables!]

(d) Using (c), if you take a **line** in $(\mathbb{P}^2)^\vee$, what does the collection of all the points on this line correspond to back in $\mathbb{P}^2$? Explain your answer carefully.

(e) The following is a set of lines in $\mathbb{P}^2$, labelled a to g.



Sketch the set of points in $(\mathbb{P}^2)^\vee$ dual to these lines, and label them A to G **corresponding to the similarly named lines**. Make sure that your sketch reflects when three or more of the points are on a line. [Hint: in addition to the obvious places where three or more lines meet, the three vertical lines meet at infinity!! Part (c) is crucial in this problem.]

Remark 2.6. While we have been covering [CLO] we have stuck to the notation $\mathbb{I}(V)$ for the ideal associated to a variety V (or indeed to any subset V of affine or projective space). Now, however, we will convert to the more standard notation I_V.

3. Cohen-Macaulay Graded Rings

Let $R = k[x_0, \dots, x_n]$, where k is a field. The following is copied from [AM] page 106.

Definition 3.1. A *graded ring* is a ring A together with a family $([A]_n)_{n\geq 0}$ of subgroups of the additive group of A, such that $A = \bigoplus_{n\geq 0}[A]_n$ and $[A]_m[A]_n \subseteq [A]_{m+n}$ for all $m, n \geq 0$.

The following is the main example for us.

Example 3.2. $R = k[x_0, \dots, x_n]$ is a graded ring since $R = \bigoplus_{t\geq 0}[R]_t$, where $[R]_t$ is the k-vector space of homogeneous polynomials (i.e. forms) of degree t over k. Recall that

$$\dim[R]_t = \binom{t+n}{n}.$$

Notice that in particular, R is even a little more: it is a *standard graded k-algebra*, meaning that $[R]_0 = k$, the elements of R are generated by the elements of $[R]_1$, and the components are actually finite dimensional vector spaces over k.

From now on we view R as a graded ring, and focus on homogeneous ideals (cf. [CLO] Chapter 8, Section 3). For convenience let's always assume that k is an infinite field. The following definition is from [H] Exercise II.5.10. That exercise also shows its importance in the study of subschemes of projective space, although we omit this topic.

Definition 3.3. If $I \subset R$ is a homogeneous ideal then its *saturation*, I^{sat}, is defined by

$$I^{sat} = \{f \in R \mid \text{for each } 0 \leq i \leq n \text{ there is some } m_i \text{ so that } x_i^{m_i} f \in I\}.$$

The ideal I is *saturated* if $I = I^{sat}$.

Exercise 27. Prove that if I is a homogeneous ideal then so is its saturation I^{sat}.

Exercise 28. Find the saturation of each of the following ideals (or explain why it is already saturated).

(a) $\langle x^2, y^2, z^2 \rangle \subset k[x, y, z]$.

(b) $\langle x^2, y^2, z^2 \rangle \subset k[w, x, y, z]$.

(c) $\langle x^2, xy, xz \rangle \subset k[x, y, z]$.

As noted in Remark 2.6, if $V \subset \mathbb{P}^n$ is a projective subvariety (or subscheme) then we denote by I_V its homogeneous ideal. (This differs from the notation in [CLO] but the definition is the same.)

Exercise 29. Show that the ideal I_V as defined in [CLO] is a saturated ideal.

Remark 3.4. It's worth noting that when I is not of the form I_V for any subvariety (or subscheme) V, then I is not necessarily a saturated ideal, and this means that

$$\times L : [R/I]_t \to [R/I]_{t+1}$$

is not necessarily injective (Exercise 33 (c)). So the fact that the first map in the exact sequence (4.1) in Remark 4.11 is injective depends on the fact that I_V is a saturated ideal, i.e. that R/I_V has depth ≥ 1 (see Definition 3.7).

Example 3.5. Let $R = k[x, y, z]$ and $I = \langle x^2, xy, xz \rangle$. Then the Hilbert function of R/I begins with the sequence $(1, 3, 3, \dots)$ but clearly $x \in [R/I]_1$ is in the kernel of multiplication by any linear form L. Notice also that the vanishing locus of I is not zero-dimensional, as might have been suggested by the fact that the Hilbert function is equal in degrees 1 and 2, but instead consists of the line $x = 0$. In fact, even though the Hilbert function takes the same value 3 in degrees 1 and 2, the discussion after Remark 4.8 does not apply because this ideal is not I_V for any variety V. The key is that the multiplication $\times L$ in (4.1) is not an injection (why not?). We will talk more about this soon.

Exercise 30. Find the entire Hilbert function of the algebra given in Example 3.5. Is there any other degree in which $\times L$ fails to be injective? Find the saturation of this ideal. What subvariety of $\mathbb{P}^2$ corresponds to this saturation?

The following definition can be made more generally for a finitely generated graded R-module, but for our purposes it is enough to define it for standard graded k-algebras. So from now on I will be a homogeneous ideal defining a standard graded algebra R/I.

Definition 3.6. An element $F \in R/I$ of degree ≥ 1 is a *non-zerodivisor* (or sometimes *regular element*) if, for any $G \in R/I$, the condition $FG = 0$ forces $G = 0$. A *regular sequence* for R/I is a sequence of homogeneous polynomials $F_1, \dots, F_r \subset \mathfrak{m}$ such that

F_1 is a non-zerodivisor on R/I,

F_2 is a non-zerodivisor on $R/\langle I, F_1 \rangle$,

$\vdots$

F_r is a non-zerodivisor on $R/\langle I, F_1, \dots, F_{r-1} \rangle$.

Definition 3.7. The *depth* of R/I is the integer

$$\operatorname{depth}(R/I) = \sup\{j \mid \text{there is some regular sequence in } \mathfrak{m} \text{ of length } j \text{ for } R/I\}.$$

Remark 3.8. It is a fact that if a regular sequence of length m exists for R/I, then a regular sequence of length m consisting of linear forms can be found. Furthermore, in this case it suffices to choose m "sufficiently general" linear forms (once m is known). See [BH] Prop. 1.5.12.

Example 3.9. Let $R = k[x_0, x_1, x_2, x_3]$. Let C be a line in $\mathbb{P}^3$, defined by $I_C = \langle x_2, x_3 \rangle$. We claim that (x_1, x_0) is a regular sequence for R/I_C.

Notice that $R/I_C \cong k[x_0, x_1]$. If $F \in R/I_C$ is such that $x_1 F = 0$ in R/I_C then clearly $F = 0$ so x_1 is a non-zerodivisor for R/I_C. Now $R/\langle I_C, x_1\rangle \cong k[x_0]$. If $F \in R/\langle I_C, x_1\rangle$ is such that $x_0 F = 0$ in $R/\langle I_C, x_1\rangle$ then $F = 0$, so x_0 is a non-zerodivisor for $R/\langle I_C, x_1\rangle$ and we are done. In particular, $\operatorname{depth}(R/I_C) = 2$.

Example 3.10. Let $C \subset \mathbb{P} = \mathbb{P}^3_{\mathbb{R}}$ be the image of the map

$$\phi : \mathbb{P}^1 \to \mathbb{P}^3$$

given by $[s,t] \mapsto [s^3, s^2t, st^2, t^3]$ for $s, t \in \mathbb{R}$. This image is called the *twisted cubic curve* in $\mathbb{P}^3$. It is a fact that its homogeneous ideal is $\langle x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2\rangle$. Furthermore, $\operatorname{Kdim} R/I_C = 2$ (see below for the definition) and $\operatorname{depth}(R/I_C) = 2$. We will accept this as a fact.

Exercise 31. Let $R = k[x_0, x_1, x_2, x_3]$. Let V be a set of two skew lines in $\mathbb{P}^3$, say $V = \mathbb{V}(x_0, x_1) \cup \mathbb{V}(x_2, x_3)$. The homogenous ideal is $I_V = \langle x_0x_2, x_0x_3, x_1x_2, x_1x_3\rangle$ (you can just accept this as a fact).

(a) Let $L = x_0 + x_1 + x_2 + x_3$. Let $G \in R$ be a homogeneous polynomial and let $\bar{G} \in R/I_V$ be the image of G in R/I_V. If $L\bar{G} = 0$ in R/I_V show that $\bar{G} = 0$ in R/I_V (i.e. $G \in I_V$). Conclude that L is a regular element.
(b) Geometrically, L defines a plane in $\mathbb{P}^3$. Find the two points of $V \cap \mathbb{V}(L)$.
(c) Since through two distinct points of $\mathbb{P}^3$ there passes a unique line, there must be another linear form L', not a scalar multiple of L, passing through the two points you found in (b). Find one such L'.
(d) Show that $x_iL' \in \langle L, I_V\rangle = \langle L, x_0x_2, x_0x_3, x_1x_2, x_1x_3\rangle$ for all $0 \le i \le 3$.
(e) Conclude that $R/\langle L, I_V\rangle$ does not have any non-zerodivisors, so $\operatorname{depth}(R/I_V) = 1$.
(f) Note that the fact that an algebra R/I has depth ≥ 1 means that there exists a non-zerodivisor. It doesn't mean that zerodivisors don't exist. For example, find a zerodivisor for R/I_V.

The next few exercises try to draw some connections between the notion of the saturation of a homogeneous ideal I and the depth of R/I.

Exercise 32. We have noted that if I is a homogeneous ideal then so is I^{sat} (Exercise 27). Denote by $[I]_t$ the vector space of homogeneous polynomial of degree t in I. Thus we have decompositions

$$I = \bigoplus_{t \ge 1} [I]_t \quad \text{and} \quad I^{sat} = \bigoplus_{t \ge 1} [I^{sat}]_t.$$

Prove that for $t \gg 0$, $[I]_t = [I^{sat}]_t$. (Hint: use the Noetherian property.)

Exercise 33. Let $\mathfrak{m} = \langle x_0, \ldots, x_n\rangle$, the irrelevant ideal in the graded ring $R = k[x_0, \ldots, x_n]$. Let I be a homogeneous ideal. Define

$$I : \mathfrak{m} = \{f \in R \mid fm \in I \text{ for all } m \in \mathfrak{m}\}.$$

(a) Verify that $I : \mathfrak{m}$ is a homogeneous ideal in R.
(b) Show that I is saturated if and only if $I : \mathfrak{m} = I$.
(c) We define a *socle element* of R/I to be a non-zero element $f \in [R/I]_t$ (for some t) such that f is annihilated by $\mathfrak{m}$. This corresponds to an element $f \in [I : \mathfrak{m}]_t \backslash [I]_t$. In particular, f is in the kernel of $\times L : [R/I]_t \to [R/I]_{t+1}$ for all $L \in [R]_1$. Show that I is saturated if and only if R/I has no socle.

Exercise 34. Prove that if $\operatorname{depth}(R/I) \geq 1$ then I is saturated.

Remark 3.11. In the last few exercises we have shown that

I is saturated if and only if $I : \mathfrak{m} = I$ if and only if R/I has no socle.

We also saw that if $\operatorname{depth}(R/I) \geq 1$ then I is saturated. In fact the converse is true, and we have the fact that

I is saturated if and only if $\operatorname{depth}(R/I) \geq 1$.

To see the last direction, recall that the associated primes of an ideal I are the prime ideals of the form $Ann_R(f)$ for some $f \in R/I$, and consequently that I is not saturated if and only if $\mathfrak{m}$ is an associated prime (for one direction, take f to be an element of largest degree in I^{sat}/I). Now if $\operatorname{depth}(R/I) = 0$ then you can check that $\mathfrak{m}$ is an associated prime for some primary component of I (exercise), hence I is not saturated. That is, if I is saturated then $\operatorname{depth}(R/I) \geq 1$.

Remark 3.12. The following is a useful fact. Let L be a linear form and assume $\operatorname{depth}(R/I) \geq 1$ for some graded algebra R/I. Then for any t, multiplication by L gives the following exact sequence:

$$0 \to \left[\frac{I : L}{I}\right]_{t-1} \to \left[\frac{R}{I}\right]_{t-1} \xrightarrow{\times L} \left[\frac{R}{I}\right]_t \to \left[\frac{R}{\langle I, L\rangle}\right]_t \to 0 \tag{3.1}$$

(think about what the kernel of $\times L$ is), which induces a short exact sequence

$$0 \to [R/(I : L)]_{t-1} \xrightarrow{\times L} [R/I]_t \to R/\langle I, L\rangle \to 0.$$

Now assume that $\operatorname{depth}(R/I) \geq 1$ and let L be a general linear form. By Remark 3.8 we know that L is a non-zerodivisor for R/I. This means that the first term in (3.1) is zero, and we have a short exact sequence

$$0 \to [R/I]_{t-1} \xrightarrow{\times L} [R/I]_t \to [R/\langle I, L\rangle]_t \to 0.$$

More generally, in this situation we have

$$0 \to R/I(-1) \xrightarrow{\times L} R/I \to R/\langle I, L\rangle \to 0$$

is an exact sequence of graded algebras.

Definition 3.13. Let $\mathfrak{p}$ be a homogeneous prime ideal in R. The *height* of $\mathfrak{p}$ is the supremum of all integers i such that there exists a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \cdots \subsetneq \mathfrak{p}_i = \mathfrak{p}$ of homogeneous prime ideals in R. For a homogeneous ideal I, the height of I is the infimum of the heights of prime ideals in R containing I. This is the *codimension* of I.

The *Krull dimension* of R/I is the supremum of the heights of all homogeneous prime ideals in the ring R/I (not R). Equivalently, we want the longest length of a chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$$

of prime ideals in R, where $I \subset \mathfrak{p}_0$.

The geometric version of the definition of the Krull dimension is the following (cf. [H] page 5):

Definition 3.14. If X is a variety then the *dimension* of X is the supremum of all integers i such that there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_i$ of non-empty irreducible subvarieties of X.

Notation 3.15. To avoid confusion we will denote the dimension of a variety X by $\dim X$ and the Krull dimension of a graded algebra R/I by $\operatorname{Kdim}(R/I)$.

Example 3.16. (a) As one might intuitively expect, $\dim \mathbb{P}^n = n$ while

$$\operatorname{Kdim}(k[x_0, \ldots, x_n]) = n + 1.$$

Indeed, the relevant chains (thinking of an i-dimensional subspace of $\mathbb{P}^n$ as $\mathbb{P}^i$) are

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$$

and

$$\langle 0 \rangle \subset \langle x_0 \rangle \subset \langle x_0, x_1 \rangle \subset \langle x_0, x_1, x_2 \rangle \subset \cdots \subset \langle x_0, \ldots, x_{n-1} \rangle \subset \langle x_0, \ldots, x_n \rangle.$$

(b) If $\mathbb{V}(I)$ is a single point then $\dim \mathbb{V}(I) = 0$ while $\operatorname{Kdim}(R/I) = 1$.

(c) In general, let I be a homogeneous ideal. Then the Krull dimension of R/I is one more than the dimension of $\mathbb{V}(I)$.

(d) Say V is a line in $\mathbb{P}^4$ defined by the ideal $\langle x_0, x_1, x_2 \rangle$. We know that a line has dimension 1, so in $\mathbb{P}^4$ it has codimension (i.e. height) $4 - 1 = 3$. We expect the Krull dimension of R/I_V to be $1 + 1 = 2$. Let's look at the above definitions.

$$\langle 0 \rangle \subset \langle x_0 \rangle \subset \langle x_0, x_1 \rangle \subset \langle x_0, x_1, x_2 \rangle = I_V.$$

Any other prime ideal containing I_V (e.g. $\langle x_0, x_1, x_2, x_3 \rangle$) has bigger height. So the height of I_V is 3 as expected.

To get the Krull dimension of R/I_V we look for homogeneous prime ideals containing I_V. These include $\langle x_0, x_1, x_2 \rangle$, $\langle x_0, x_1, x_2, x_3 \rangle$ and $\langle x_0, x_1, x_2, x_3, x_4 \rangle$. You can convince yourself that this means that the Krull dimension of R/I_V is 2.

To get the dimension of V we look at chains of non-empty irreducible subvarieties. We get $Z_0 \subsetneq Z_1$ where Z_0 is a single point and $Z_1 = V$. These correspond to the first two ideals in the previous chain.

Remark 3.17. (a) One can show that

$$\text{height } I + \operatorname{Kdim}(R/I) = \dim R = n + 1.$$

(b) It is always the case that $\operatorname{depth}(R/I) \leq \operatorname{Kdim}(R/I)$.

Definition 3.18. The algebra R/I is *Cohen-Macaulay* if $\operatorname{depth}(R/I) = \operatorname{Kdim}(R/I)$. If V is a subvariety of $\mathbb{P}^n$ with saturated homogenous ideal I_V, and if R/I_V is Cohen-Macaulay, then V is said to be *arithmetically Cohen-Macaulay*, sometimes denoted ACM.

Example 3.19. Example 3.9 and Example 3.10 show that a line and a twisted cubic are both ACM curves in $\mathbb{P}^3$.

Exercise 35.

(a) R itself is Cohen-Macaulay.

(b) If $I = I_V$ where $\dim V = 0$ then R/I is Cohen-Macaulay (i.e. V is ACM). In other words, a finite set of points in $\mathbb{P}^n$ is always ACM.

(c) The same does not hold for higher dimension. In particular, find a curve C for which R/I_C is not Cohen-Macaulay (i.e. C is not ACM). (Hint: see Exercise 31.)

Remark 3.20. Let V be a subvariety of $\mathbb{P}^n$. Let I_V be its saturated homogeneous ideal. If the number of minimal generators of I_V is equal to $n - \dim V$ (i.e. equal to the *codimension* of V in $\mathbb{P}^n$) then V is called a *complete intersection* and is automatically arithmetically Cohen-Macaulay. The minimal generators of I_V then form a regular sequence in $\mathfrak{m}$.

The following definition gives a very important class of Cohen-Macaulay algebras. We will not prove the equivalence of the conditions.

Definition 3.21. Let I be a homogenous ideal in R. Then R/I is *artinian* if any of the following equivalent conditions holds.

(a) R/I is finite dimensional as a k-vector space.
(b) $\operatorname{Kdim}(R/I) = 0$.
(c) If $\mathfrak{m}$ is the irrelevant ideal of R/I then $\mathfrak{m}^p = 0$ in R/I for some (hence all sufficiently large) $p \geq 1$, i.e. (viewing $\mathfrak{m}$ as the irrelevant ideal of R), $\mathfrak{m}^p \subset I$ for some $p \geq 1$.
(d) For each $0 \leq i \leq n$ there is some integer p_i such that $x_i^{p_i} \in I$.
(e) For sufficiently large d we have $[I]_d = [R]_d$.
(f) If k is algebraically closed, a sixth equivalent condition is $\mathbb{V}(I) = \emptyset$.
(g) R/I satisfies the *descending chain condition* for ideals.

Remark 3.22. (a) Assume that $\operatorname{depth}(R/I) \geq 1$ and $\operatorname{Kdim}(R/I) \geq 1$. Let L be a general linear form (hence a non-zerodivisor on R/I). Then $\operatorname{depth}(R/\langle I, L\rangle) = \operatorname{depth}(R/I) - 1$ and $\operatorname{Kdim}(R/\langle I, L\rangle) = \operatorname{Kdim}(R/I) - 1$.

(b) Of course if R/I is artinian then it is Cohen-Macaulay since

$$0 \leq \operatorname{depth}\ (R/I) \leq \operatorname{Kdim}(R/I) \ = \ 0.$$

Given a Cohen-Macaulay algebra, we construct from it an artinian algebra as follows.

Proposition 3.23. *Let R/I be a graded Cohen-Macaulay algebra of depth = Krull dimension = d. Let $L_1, \dots, L_d$ be a regular sequence of linear forms. Then $R/\langle I, L_1, \dots, L_d\rangle$ is an artinian graded algebra. If the L_i are sufficiently general, this is called the* general artinian reduction *of R/I.*

4. Introduction to Hilbert functions

4.1. **Graded modules.** The notion of an *R-module* generalizes that of a k-vector space. The following definition is copied from [AM] page 17, where you can read more about the subject.

Definition 4.1. Let A be a ring. An *A-module* is an abelian group M (written additively) on which A acts linearly. More precisely, it is a pair (M, μ) where M is an abelian group and μ is a mapping of $A \times M$ into M such that, if we write ax for $\mu(a, x)$ where $a \in A$ and $x \in M$, we have

$$a(x + y) = ax + ay,$$
$$(a + b)x = ax + bx,$$
$$(ab)x = a(bx),$$
$$1x = x$$

for all $a, b \in A$ and $x, y \in M$.

Example 4.2.
1. If $A = k$, a field, then the notions of A-module and k-vector space coincide.
2. If $M = I$ is an ideal of A then M is an A-module. In particular, A itself is an A-module.
3. If $A = \mathbb{Z}$ then the notions of A-module and abelian group coincide, where we define
$$nx = \underbrace{x + \cdots + x}_{n \text{ times}}$$
for $n \geq 1$.

The following is copied from [AM] page 106.

Definition 4.3. If A is a graded ring (see Definition 3.1), a *graded module* is an A-module M together with a family $([M]_t)_{t\in\mathbb{Z}}$ of subgroups of M such that
$$M = \bigoplus_{t\in\mathbb{Z}} [M]_t \quad \text{and} \quad [A]_m[M]_t \subset [M]_{m+t} \text{ for all } m, t \in \mathbb{Z}.$$

If $f \in R$ is any polynomial, we can always decompose f as a sum of its homogeneous components
$$f = f_0 + f_1 + \cdots + f_d.$$
By linearity, to understand fm for $m \in M$, it's enough to understand how homogeneous polynomials act on *homogenenous* elements $m \in M$. But again by linearity, it's enough to understand how linear forms act, and in fact it's enough to understand $x_0m, \ldots, x_nm$.

Example 4.4. Each of the following is a graded R-module.

1. $R = k[x_0, \ldots, x_n]$ is also a graded R-module.
2. The shifted module $R(m)$ is defined by $[R(m)]_t = [R]_{m+t}$.
3. If I is a homogeneous ideal then R/I is a graded R-module. Recall that to stress that the components $[R/I]_t$ are vector spaces, we often refer to R/I as a graded algebra rather than a graded ring.
4. Let $R = k[w, x, y, z]$ and $I = \langle w, x, y, z^2\rangle$. Then
$$\dim[R/I]_t = \begin{cases} 1 & \text{if } t = 0, 1; \\ 0 & \text{if } t \neq 0, 1. \end{cases}$$
The behavior of multiplication for R/I by a linear form is inherited from R modulo $\langle w, x, y, z^2\rangle$.
5. Let $R = k[w, x, y, z]$. Let M be a graded module defined as follows: $\dim[M]_t = 2$ for $t = 0, 1$ and $[M]_t = 0$ otherwise. Assume that we have chosen bases for $[M]_0$ and $[M]_1$. Let
$$A = \begin{bmatrix} a & 2b \\ 3c & 4d \end{bmatrix}$$
where $a, b, c, d \in k$. If $L = aw + bx + cy + dz$ and
$$m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \in [M]_0$$
then we define
$$Lm = A \cdot m$$

where the latter is the matrix product, viewed as an element of $[M]_1$. This determines the module structure of M.

Finally, if M is a graded R-module, we define the *annihilator* of M to be the ideal

$$\mathrm{Ann}(M) = [0 : M] = \{f \in R \mid fm = 0 \text{ for all } m \in M\}.$$

This is in fact a homogeneous ideal (since M is graded).

4.2. **Hilbert functions and Hilbert polynomials.** Let M be a graded R-module, so we also have

$$M = \bigoplus_{t \in \mathbb{Z}} [M]_t$$

where $[M]_t$ is the degree t component of M. We define the *Hilbert function* of M to be the function

$$h_M : \mathbb{Z} \to \mathbb{Z}^{\geq 0}$$

given by $h_M(t) = \dim_k [M]_t$. The *Hilbert polynomial* of M is the polynomial $p_M(t)$ defined by the following result.

Theorem 4.5 (Hilbert-Serre). *Let M be a finitely generated graded R-module. Then there is a unique polynomial $p_M(t) \in \mathbb{Q}[t]$ such that $p_M(t) = h_M(t)$ for all $t \gg 0$. Furthermore,* $\deg p_M(t) = \dim Z(\mathrm{Ann}(M))$*, where Z denotes the vanishing locus of a homogeneous ideal.*

Proof. See [H] Theorem I.7.5. □

For us the main situation will be when $M = R/I$ is a standard graded k-algebra (see the definition in Example 3.2), where I is a homogeneous ideal. If $I = I_V$ for some subvariety (or subscheme) $V \subset \mathbb{P}^n$ then we will sometimes write $h_V(t)$ for $h_{R/I_V}(t)$, and $p_V(t)$ for the corresponding Hilbert polynomial.

Remark 4.6. We will sometimes be interested in the first difference of the Hilbert function, which is defined as the function

$$\Delta h_{R/I}(t) = h_{R/I}(t) - h_{R/I}(t-1)$$

for all t. Inductively we also define $\Delta^2 h_{R/I}(t)$, $\Delta^3 h_{R/I}(t)$, etc.

Remark 4.7. First let's see what general facts we can say immediately about the Hilbert function $h_{R/I}(t)$.

1. $h_{R/I}(t) = 0$ for $t < 0$ and $h_{R/I}(0) = 1$.
2. If $I = I_V$ for some subvariety (or subscheme) $V \subset \mathbb{P}^n$ then $\deg(p_V(t)) = \dim V$ thanks to Theorem 4.5.
3. If $I = I_V$ for some subvariety (or subscheme) $V \subset \mathbb{P}^n$ then I_V is saturated, so depth $(R/I_V) \geq 1$ (Exercise 34). Thus a general linear form is a non-zerodivisor (Remark 3.8). This gives the injective homomorphism

$$\times L : [R/I_V]_t \to [R/I_V]_{t+1}.$$

As a consequence, we have that $h_V(t) \leq h_V(t+1)$ for all t (in particular, for all $t \geq 0$).

4. Assume that $I = I_V$ for some subvariety (or subscheme) $V \subset \mathbb{P}^n$ of dimension d, so the Hilbert polynomial of V has the form
$$p_V = a_d x^d + (\text{terms involving lower powers of } x).$$
Then $a_d \cdot d!$ is an invariant of V called its *degree*.

Exercise 36. Let $R = k[x, y]$ and let $I = \langle x^4, x^2y^3, xy^4, y^6 \rangle$.

(a) Draw a picture, using the integer points in the first quadrant and shading, to represent the monomials in I.
(b) What are the monomials *not* in I? (I want the complete list.)
(c) What is the Hilbert function of R/I?
(d) What is the Hilbert polynomial of R/I?

Remark 4.8.

1. In Remark 3.20 we defined a special kind of variety called a *complete intersection*. it turns out that for a complete intersection V, the degree of V is the product of the degrees of the minimal generators of I_V.
2. If V is a finite set of points (0-dimensional), its Hilbert polynomial is a constant (degree 0 polynomial) that is equal to the number of points of V. See Exercise 4 and Exercise 41.

A truly amazing fact is that we know all possible Hilbert functions of standard graded algebras! (The challenge is to derive useful information from this knowledge!) This is provided by Macaulay's theorem. We recall this now, without proof.

Definition 4.9. Let m and d be positive integers. The *d-binomial expansion of m* is the expression
$$m = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_j}{j}$$
where $a_d > a_{d-1} > \cdots > a_j \geq j \geq 1$. We further define
$$m^{(d)} = \binom{a_d + 1}{d+1} + \binom{a_{d-1} + 1}{d} + \cdots + \binom{a_j + 1}{j+1}.$$

The blockbuster result we now quote is the following:

Theorem 4.10 (Macaulay [Mac])**.** *Let $\underline{h} = (1, h_1, h_2, \dots)$ be a sequence of positive integers. Then $\underline{h}$ is the Hilbert function of some standard graded algebra R/I, where $R = k[x_1, \dots, x_n]$, $n = h_1$ and k is a field, if and only if $h_{i+1} \leq h_i^{(i)}$ for all $i \geq 0$.*

A sequence satisfying this property is called an O-sequence.

Exercise 37. Is the following an O-sequence?
$$(1, 5, 12, 17, 25, 36)$$

Remark 4.11. There are two natural directions to go at this point. First, if you know things about V, what can you say about what the Hilbert function h_V looks like? For example, if V is a finite set of points then we know that h_V is eventually a polynomial of degree 0, i.e. a constant. More interesting in some sense is the second direction: if you know something unusual about h_V, what does that tell you about V? There are several ways of obtaining geometric information about a variety from knowledge of its Hilbert function.

See for instance [D], [BGM], [CM]. We omit details. But let's start with more elementary observations.

To illustrate how $h_V(t)$ can give information about a *variety* V, suppose we know that for some t_0 we have $h_V(t_0) = h_V(t_0+1)$ (i.e. for some t_0 we have equality in item 3 of Remark 4.7). We claim that this forces V to be zero-dimensional.

Indeed, consider the exact sequence (see Remark 3.12)

$$0 \to [R/I_V]_t \xrightarrow{\times L} [R/I_V]_{t+1} \to [R/\langle I_V, L\rangle]_{t+1} \to 0 \tag{4.1}$$

for any t (the injectivity comes because I_V is saturated – see Exercise 29). It follows that $\Delta h_V(t)$ is the Hilbert function of $R/\langle I_V, L\rangle$, which is a standard graded algebra (it is generated in degree 0 only – see Example 3.2). Thus if $h_V(t_0) = h_V(t_0+1)$, this means that $\Delta h_V(t_0+1) = 0$, so the component of $R/\langle I_V, L\rangle$ in degree t_0+1 is zero. Hence $R/\langle I_V, L\rangle$ is zero in all degrees $\geq t_0+1$, so the Hilbert polynomial of $R/\langle I_V, L\rangle$ is the zero polynomial. This means that $h_V(t) = h_V(t+1)$ for all $t \geq t_0$, so $p_V(t)$ is a constant polynomial. Then by Theorem 4.5, V is zero-dimensional.

Remark 4.12. Remark 4.11 says, in particular, that the stated assumption about the Hilbert function forces V to be a finite set of points and p_V to be a constant polynomial. We now give an interpretation of this constant. So assume that V is a finite set of points. We claim that the number of points of V is the value $h_V(t)$ for all $t \gg 0$. (In fact for all $t \geq t_0$ where t_0 is as in Remark 4.11.)

Our proof will be by induction on the number of points. If $|V| = 1$, we can write $I_V = \langle x_1, \dots, x_n\rangle$ so $R/I_V \cong k[x_0]$, and the value of the Hilbert function is 1 in all degrees ≥ 0. Now let V' be a set of d points, P a single point distinct from any of the points of V', and $V = V' \cup P$. Of course we have $[I_V]_t \subseteq [I_{V'}]_t$ for all t. We have the exact sequence

$$0 \to [I_{V'}]_t/[I_V]_t \to [R/I_V]_t \to [R/I_{V'}]_t \to 0$$

for $t \gg 0$. By induction, the third vector space in this sequence has dimension d, so it is enough to check that for $t \gg 0$ the first has dimension 1. In fact, we'll show that for any t it has dimension either 0 or 1, with the latter value for $t \gg 0$.

If we set $N = \binom{t+n}{n}$, we have seen that $\dim[R]_t = N$, so a typical element of $[R]_t$ has the form

$$F = a_1 x_0^t + \cdots + a_N x_n^t.$$

Letting $Q = [q_0, \dots, q_n]$ be any point of V, we see that F vanishes at Q if and only if

$$a_1 q_0^t + \cdots + a_N q_N^t = 0.$$

This is a homogeneous linear equation in the variables $a_1, \dots, a_N$. So in our situation, F vanishing at the points of V' (i.e. $F \in [I_{V'}]_t$) means we have a homogeneous linear system of d equations. Furthermore, F also vanishing at P (i.e. $F \in [I_V]_t$) adds one more homogeneous linear equation to the system. So either the new equation is a linear combination of the d previous equations (in which case $\dim[I_{V'}]_t/[I_V]_t = 0$) or else it imposes one new condition (meaning $\dim[I_{V'}]_t/[I_V]_t = 1$). If $t \gg 0$, it is not hard to construct a hypersurface of degree t (e.g. a union of hyperplanes) vanishing on V' but not on P, so not all solutions of the first d equations also solve the $(d+1)$-st equation, and the quotient is 1-dimensional.

5. The connection to minimal free resolutions

In this section we define several kinds of algebras (including a second view of Cohen-Macaulay algebras) in terms of the minimal free resolution

$$0 \to \mathbb{F}_r \to \mathbb{F}_{r-1} \to \cdots \to \mathbb{F}_1 \to R \to R/I \to 0,$$

where $R = k[x_1, \dots, x_n]$ is the coordinate ring for $\mathbb{P}^{n-1}$. The *projective dimension* proj dimR/I is the integer r in this minimal free resolution.

1. **Cohen-Macaulay algebras.** The following conditions are equivalent.
   (a) R/I is Cohen-Macaulay with depth = Krull dimension = d. (Recall that if I defines a variety V in $\mathbb{P}^{n-1}$ then $\dim V = d - 1$.)
   (b) The projective dimension r in the minimal free resolution satisfies $r = n - d$. In the special case where $I = I_V$ for some projective variety V, we have

   $$r = n - d = (n-1) + 1 - (\dim V + 1) = \text{codimension of } V \text{ in } \mathbb{P}^{n-1}.$$

   Again, if $I = I_V$ for a projective variety V then we say V is *arithmetically Cohen-Macaulay (ACM)* if R/I_V is Cohen-Macaulay.

   Assume R/I has Krull dimension d (and temporarily we do not assume that R/I is Cohen-Macaulay). The *canonical module* of R/I is

   $$K_{R/I} = \text{Ext}_R^{n-d}(R/I, R)(-n).$$

   When R/I is Cohen-Macaulay, the minimal free resolution of $K_{R/I}$ is the dual of the minimal free resolution of R/I.
2. **Gorenstein algebras.** R/I is *Gorenstein* if it is Cohen-Macaulay (i.e. $r = n - d$) and the rank of $\mathbb{F}_r$ is 1. If $I = I_V$ for a projective variety V then we say that V is *arithmetically Gorenstein (AG)*.
3. **Complete Intersections.** R/I is a *complete intersection* if the rank of $\mathbb{F}_1$ (which is equal to the number of minimal generators of I) is equal to the codimension of $\mathbb{V}(I)$ in $\mathbb{P}^n$.

   The minimal free resolution of a complete intersection is given by the *Koszul resolution*, which is the following. Let $I = (F_1, \dots, F_r)$ be a regular sequence (i.e. the ideal of a complete intersection), with $d_i = \deg F_i$. Then we have the following minimal free resolution for R/I:

   $$0 \to \mathbb{F}_r \to \mathbb{F}_{r-1} \to \cdots \to \mathbb{F}_1 \to R \to R/I \to 0$$

where

$$\begin{aligned}
F_1 &= \bigoplus_{i=1}^{r} R(-d_i) \\
F_2 &= \bigwedge^2 \mathbb{F}_1 &&= \bigoplus_{1\le i_1<i_2\le r} R(-d_{i_1}-d_{i_2}) \\
F_3 &= \bigwedge^3 \mathbb{F}_1 &&= \bigoplus_{1\le i_1<i_2<i_3\le r} R(-d_{i_1}-d_{i_2}-d_{i_3}) \\
&\vdots \\
F_r &= \bigwedge^r \mathbb{F}_1 &&= R(-d_1-\dots-d_r)
\end{aligned}$$

In particular, a complete intersection is Gorenstein, and (hence) Cohen-Macaulay. If $V \subset \mathbb{P}^n$ is a projective variety with homogenous ideal I_V satisfying the above condition then we also say that V itself is a complete intersection.

4. **Level algebras.** R/I is *level* if it is Cohen-Macaulay (i.e. the projective dimension $r = n - d$) and the direct summands of $\mathbb{F}_r$ all have the same twist: $\mathbb{F}_r = \bigoplus R(-m)$ for a fixed m.

We also mention the **Auslander-Buchsbaum formula**. In the setting of standard graded algebras R/I, where $R = k[x_0, \dots, x_n]$, we have

$$\text{proj dim } R/I + \text{depth } R/I = n + 1.$$

For example, if I is the homogeneous ideal of an ACM curve in $\mathbb{P}^3$ then the minimal free resolution of R/I has the form

$$0 \to \mathbb{F}_2 \to \mathbb{F}_1 \to R \to R/I \to 0$$

so the projective dimension is 2. On the other hand, being ACM we have that the depth is equal to the Krull dimension, and being a curve means that the Krull dimension is 2. Thus the depth is 2, and we have

$$2 + 2 = 3 + 1 = 4$$

while if I is the homogeneous ideal of a non-ACM curve in $\mathbb{P}^3$, the projective dimension increases by 1 and the depth drops by 1, so we have

$$3 + 1 = 3 + 1 = 4.$$

(The Auslander-Buchsbaum formula actually applies to more general situations, but we won't go into that here.)

Exercise 38. Show that if $R = k[x, y]$, where k is a field, and if R/I is artinian and Gorenstein then in fact R/I is a complete intersection. We will see examples to show that this is no longer true in three or more variables.

Remark 5.1. If R/I is Gorenstein, it is isomorphic to a twist of its canonical module. This implies:

The Hilbert function of an artinian Gorenstein algebra is symmetric.

It also means that if you dualize the minimal free resolution, up to twist the result that you get is the same as the original resolution! In particular, the ranks of the free modules in the resolution are symmetric. For example, if R/I is a complete intersection of forms of degree 5 in 6 variables then the minimal free resolution (from the Koszul resolution) is

$$0 \to R(-30) \to R(-25)^6 \to R(-20)^{15} \to R(-15)^{20} \to R(-10)^{15} \to R(-5)^6 \to R \to R/I \to 0$$

so you can see, looking left to right and looking right to left, the symmetry of the ranks.

A consequence of this (with a little calculation) is that up to twist, R/I is self-dual. In particular, the h-vector of R/I is also symmetric. We will see that the h-vector is not necessarily unimodal, though.

The fact that R/I is self-dual (up to twist) is very useful in the study of Lefschetz properties for Gorenstein artinian algebras, as we will see.

6. Examples of Cohen-Macaulayness

Exercise 39. Play with a computer algebra program. For example, verify that five random points in $\mathbb{P}^3$ (or $n+2$ random points in $\mathbb{P}^n$) are arithmetically Gorenstein. In particular, verify that not all Gorenstein algebras are complete intersections.

Exercise 40. Check that a set of two skew lines in $\mathbb{P}^3$ is not ACM, although either line by itself is ACM.

Exercise 41. Prove that the Hilbert function of a set of d points in $\mathbb{P}^n$ is strictly increasing until it reaches the value d, at which time it becomes constant. Thus the Hilbert polynomial of a finite set of points is the constant polynomial equal to the number of points in the set.

Example 6.1. Here is an interesting variety that turns out to always be ACM. We refer to [GHM] for details. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$, i.e. it is a union of, say, r hyperplanes in $\mathbb{P}^n$. Fix an integer c with $2 \leq c \leq n$ and assume $r \geq c$. We make the special assumption that any $c+1$ of the hyperplanes meet in codimension $c+1$. (If $c = n$, this means that no $c+1$ of the hyperplanes have a common point.) For example, if $c = 2$ and $n = 3$ we have a union of r planes in $\mathbb{P}^3$ and we are assuming that no three share a line.

Notice that the special assumption also means that for $c \leq n$, any c of the hyperplanes meet in a linear variety of codimension c, and that two different choices of c of the hyperplanes give different codimension c linear varieties.

Now let V be the union of the codimension c linear varieties obtained in this way. Notice that $\deg V = \binom{r}{c}$. It turns out that V is always ACM. The main tool to prove this is a construction from liaison theory called *basic double linkage*, which is beyond the scope of these notes. See [Mi2] for details.

Three directions that have been taken in the literature to extend this example are the following. First, one can move from hyperplane arrangements to hypersurface arrangements. Second, one can relax the assumption that no $c+1$ of the hyperplanes meet in codimension c. And third, in the case that $c = 2$, we can relate this to Jacobian ideals. In all three situations, the Cohen-Macaulay question is of great interest and partial results have been obtained.

Example 6.2. Let C and C' be ACM curves in $\mathbb{P}^3$ such that $X = C \cup C'$ is a complete intersection, say of a surface of degree a and a surface of degree b. Assume that C and C' meet in a finite set of points, Y. We will sketch the proof that Y is AG. We use some machinery that is not assumed for this course, and it is not important if you do not follow the details of the argument. The point is to give an example of a nice connection between ACM varieties of some codimension (related in a strong way) and a resulting AG variety of codimension one greater.

Since C and C' are ACM of codimension 2, their minimal free resolutions have the form

$$0 \to \mathbb{F}_2 \to \mathbb{F}_1 \to I_C \to 0$$

and

$$0 \to \mathbb{G}_2 \to \mathbb{G}_1 \to I_{C'} \to 0.$$

There is a standard exact sequence

$$0 \to I_C \cap I_{C'} \to I_C \oplus I_{C'} \to I_C + I_{C'} \to 0.$$

Now, $I_C \cap I_{C'} = I_X$ and I_Y is the saturation of $I_C + I_{C'}$.

There is a process called *sheafification* that converts graded modules to sheaves, and it respects short exact sequences. We get, for any integer t, the exact sequence

$$0 \to \mathcal{I}_X(t) \to \mathcal{I}_C(t) \oplus \mathcal{I}_{C'}(t) \to \mathcal{I}_Y(t) \to 0.$$

Taking cohomology we would get long exact sequence at this point, but since X is ACM it turns out that $h^1(\mathcal{I}_X(t)) = 0$ for all t, so in fact (taking a direct sum over all t) we have a short exact sequence of saturated homogeneous ideals

$$0 \to I_X \to I_C \oplus I_{C'} \to I_Y \to 0.$$

We know the minimal free resolution for I_X from the Koszul resolution, and we wrote the minimal free resolutions for I_C and $I_{C'}$ above, so we have

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R(-a-b) & & \mathbb{F}_2 \oplus \mathbb{G}_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R(-a) \oplus R(-b) & & \mathbb{F}_1 \oplus \mathbb{G}_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & I_X & \to & I_C \oplus I_{C'} & \to & I_Y & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

There are induced horizontal maps, and applying the construction of the *mapping cone*, one obtains the free resolution

$$0 \to R(-a-b) \to R(-a) \oplus R(-b) \oplus \mathbb{F}_2 \oplus \mathbb{G}_2 \to \mathbb{F}_1 \oplus \mathbb{G}_1 \to I_Y \to 0.$$

This resolution is not minimal, but the fact that at the end we have only rank 1 (namely $R(-a-b)$ and the fact that Y has codimension 3 means that Y is not only ACM but in fact also AG.

7. Artinian reductions and h-vectors

We now explore the relation between the Hilbert function of a graded Cohen-Macaulay algebra and that of any artinian reduction. See Proposition 3.23.

Proposition 7.1. *Let $R = k[x_0, \dots, x_n]$ and let I be a homogeneous ideal of R.*

(a) Assume that I is saturated and that L is a general linear form. Then the Hilbert function of $R/\langle I, L\rangle$ is $\Delta h_{R/I}$.

(b) Let $A = R/I$ be a Cohen-Macaulay algebra of Krull dimension d. Let $L_1, \dots, L_d$ be a regular sequence of linear forms for A and let $B = A/(L_1, \dots, L_d)A$ be the corresponding artinian reduction. Then the Hilbert function of B is

$$h_B(t) = \Delta^d h_A(t).$$

It takes the value zero for all $t \gg 0$.

Proof. The assumptions of (a) imply that L is a non-zerodivisor for R/I (Exercise 34 and Remark 3.8). Then as in (4.1), we have the exact sequence

$$0 \to [R/I]_t \xrightarrow{\times L} [R/I]_{t+1} \to [R/\langle I, L\rangle]_{t+1} \to 0$$

from which the result follows by exactness. This proves (a).

For (b), since R/I is Cohen-Macaulay and $L_1, \dots, L_d$ is a regular sequence for R/I, we have for any $1 \le i \le d$ a short exact sequence

(7.1)

$$0 \to [R/\langle I, L_1, \dots, L_{i-1}\rangle]_{t-1} \xrightarrow{\times L_i} [R/\langle I, L_1, \dots, L_{i-1}\rangle]_t \to [R/\langle I, L_1, \dots, L_{i-1}, L_i\rangle]_t \to 0$$

(where the case $i = 1$ refers to the homomorphism $[R/I]_t \xrightarrow{\times L_1} [R/I]_{t+1}$). Then the result follows by induction on d and again by exactness of this sequence. The fact that it is eventually zero comes from the fact that artinian algebras are finite dimensional vector spaces over k. □

Definition 7.2. Let $A = R/I$ be a Cohen-Macaulay algebra of Krull dimension d, let $L_1, \dots, L_d$ be a regular sequence of linear forms for A and let $B = A/(L_1, \dots, L_d)A$ be the corresponding artinian reduction as in Proposition 7.1. Ignoring zero values, the Hilbert function $h_B(t) = \Delta^d h_A(t)$ is called the *h-vector* of V.

Example 7.3.

1. Let V be a set of 12 general points in the plane. Then the Hilbert function of V is

$$h_V = (1, 3, 6, 10, 12, 12, \dots)$$

so the h-vector is $(1, 2, 3, 4, 2)$.

2. Let V be the twisted cubic curve in $\mathbb{P}^3$. Then the Hilbert function of V is

$$h_V = (1, 4, 7, 10, \dots).$$

We know that V is ACM (Example 3.19) with $d = \text{Kdim}(R/I_V) = 2$. Thus by Proposition 7.1 we get

$$\Delta h_V = (1, 3, 3, 3, \dots)$$

and the h-vector is $\Delta^2 h_V = (1, 2)$.

3. Let V be the complete intersection in $\mathbb{P}^4$ of a quadric hypersurface and a cubic hypersurface. This illustrates Remark 3.20 and Remark 4.8. Then the saturated homogeneous ideal of V has two minimal generators, one of degree 2 and one of degree 3. These two polynomials form a regular sequence (since the codimension is equal to the number of generators). It can be shown that the Hilbert polynomial of V is $p_V = 3t^2 + 2$ so the degree is $3 \cdot 2! = 6 = 2 \cdot 3$ (the latter being the product of the degrees of the minimal generators). In fact the Hilbert function is

$$h_V = (1, 5, 14, 29, 50, 77, \dots)$$

so

$$\Delta h_V = (1, 4, 9, 15, 21, 27, \dots), \quad \Delta^2 h_V = (1, 3, 5, 6, 6, 6, \dots), \quad \Delta^3 h_V = (1, 2, 2, 1)$$

and this latter is the h-vector.

4. Let C be a line in $\mathbb{P}^3$. Let's find its Hilbert function h_C in a more geometric way.
   (a) Certainly $\dim[I_C]_0 = 0$ so
   $$h_C(0) = \dim[R]_0 - \dim[I_C]_0 = 1 - 0 = 1.$$
   (b) There are two independent linear forms containing C (since a line is the intersection of two planes in $\mathbb{P}^3$). So $\dim[I_C]_1 = 2$ and
   $$h_C(1) = \dim[R]_1 - \dim[I_C]_1 = 4 - 2 = 2.$$
   (c) Let $t \geq 2$. Choose any $t+1$ points, $P_1, \dots, P_{t+1}$ of C. Verify the following facts.
      (i) If F is a homogeneous polynomial of degree t vanishing at $P_1, \dots, P_{t+1}$ then F vanishes on all of C.
      (ii) There exists F homogeneous of degree t vanishing on any t of the points $P_1, \dots, P_{t+1}$ but not vanishing on all of C. (Think of unions of planes.)

   It follows from these two facts that C *imposes* $t+1$ *independent conditions* on forms of degree t. Thus
   $$h_C(t) = \dim[R]_t - \dim[I_C]_t = \dim[R]_t - (\dim[R]_t - (t+1)) = t + 1.$$

   So the Hilbert function of C is $(1, 2, 3, 4, \dots)$.

Exercise 42. (a) Prove that five points in $\mathbb{P}^2$ fail to impose independent conditions on plane cubics (i.e. forms of degree 3 in $\mathbb{C}[x_0, x_1, x_2]$) if and only if they all lie on a line.

(b) If V is a set of seven points lying on an irreducible conic in $\mathbb{P}^2$, prove that its Hilbert function is the sequence $(1, 3, 5, 7, 7, 7, \dots)$. (Hint: you can use without proof the fact that it's impossible to have three collinear points on an irreducible conic.)

(c) Describe what a set of points would look like if its Hilbert function is

$$(1, 3, 5, 6, 7, 7, 7, \dots).$$

(Hint: I would start by seeing what the "5" tells you; you can use the result of (b) even if you didn't solve it.)

Exercise 43. Let C be a set of two skew lines in $\mathbb{P}^3$, which we have seen is not ACM (Exercise 31). Without loss of generality assume that $R = k[w, x, y, z]$ and $C = \mathbb{V}(w, x) \cup \mathbb{V}(y, z)$. It happens to be true that $I_C = \langle wy, wz, xy, xz \rangle$, and you can use this fact without proof.

(a) Find the Hilbert function of R/I_C.

(b) The Krull dimension of R/I_C is 2. What is $\Delta^2 h_C$?

Remark 7.4. Note that if R/I is CM then its Hilbert function can be computed from that of the general artinian reduction by "integrating." For instance, in Example 7.3 (3) above, starting from the h-vector we could work backwards to obtain

$$\begin{array}{l}(1, 2, 2, 1)\\ (1, 1+2, 1+2+2, 1+2+2+1, 1+2+2+1+0, \dots) = (1, 3, 5, 6, 6, \dots)\\ (1, 4, 9, 15, 21, 27, \dots)\\ (1, 5, 14, 29, 50, 77, \dots)\end{array}$$

In fact, if V is ACM then its degree can be gotten simply by adding the entries of the h-vector.

Exercise 44. If V is a finite set of points with h-vector $(1, a_1, a_2, \dots, a_d)$, show that the number of points of V is $1 + a_1 + \cdots + a_d$. (Hint: see Remarks 4.11 and 7.4.)

Exercise 45. All of these calculations depend on the assumption that V is arithmetically Cohen-Macaulay, i.e. that R/I_V is a Cohen-Macaulay ring. Why?

Exercise 46. Suppose that V is an ACM surface (i.e. 2-dimensional) in $\mathbb{P}^6$ with h-vector $(1, 4, 7, 8, 2)$. Find the degree of V and find the Hilbert function of V (as a sequence, not necessarily in closed form).

Exercise 47. Let $R = k[w, x, y, z]$ and suppose $I \subset R$ is a homogeneous ideal with Hilbert function

$$h_{R/I}(t) = (1, 4, 3, 4, 5, \dots).$$

Prove that I is not saturated, and describe geometrically the saturation I^{sat} of I, and find its Hilbert function. (Hint: See Example 7.3 (4).)

8. Lefschetz Properties

In studying the depth of R/I we saw that it involves the injectivity of the multiplication $\times L$, where L is a linear form. (See Remark 3.12.) Notice that the next best thing to injectivity is surjectivity, and for some algebras R/I it can happen that for a general linear form L, the multiplication $\times L : [R/I]_t \to [R/I]_{t+1}$ is *not* always injective (i.e. the depth of R/I is zero), but $\times L$ *is* either injective or surjective for each t (in fact it is injective up to a certain degree and then surjective for each degree after that). This certainly is not true for all algebras, as we will see, and our focus will be on figuring out for which algebras R/I this desirable property actually does hold.

Definition 8.1. A graded algebra R/I has the *Weak Lefschetz Property (WLP)* if, for a general linear form L, the homomorphism defined by the multiplication $\times L : [R/I]_{t-1} \to [R/I]_t$ has *maximal rank* for all t. It has the *Strong Lefschetz Property (SLP)* if $\times L^d : [R/I]_{t-d} \to [R/I]_t$ has maximal rank for all t and all d.

Good general references for the Lefschetz properties are [MN1] and [HMMNWW]. The former is attached to these notes.

Remark 8.2. We have defined the WLP and SLP for standard graded algebras R/I, but indeed the exact same definitions apply if we replace R/I by any finite length graded R-module M. We will stick with the more restricted definition since that is the most studied situation, but see also [Mi1], [FFP], [Mar1], [Mar2] and [Mar3].

The following are some specific kinds of algebras R/I that have been the focus of research by different authors. You will see many more in the parallel courses by Pedro Macias Marques and Alexandra Seceleanu. The references that I've given for each topic are far from complete, and should just get you started if you pursue any of these directions. We will only talk about a couple of these, and not in a comprehensive way.

1. Complete intersections
   Some references: [HMNW], [MN2], [I], [AR], [BMMN1], [BMMN2];
2. Gorenstein algebras
   Some references: [Ik], [B], [BMMNZ2], [G], [GZ], [AAISY];
3. Ideals generated by powers of linear forms
   Some references: [SS], [MMN2], [HSS], [MM], [BL], [MN3], [MT], [HMNT], [POLITUS];
4. Monomial ideals and ideals coming from combinatorics in different ways.
   Some references: [BMMNZ1], [MMN1], [AB], [AL], [CN], [CJMN].

Remark 8.3. It is important to notice that as L ranges over $[R]_1$, the rank of $\times L$ is lower semicontinuous, meaning that there is an open set where it achieves the greatest rank among all such L, and special L could have lower ranks. Thus to prove that R/I has WLP or SLP, it is enough to find *one* linear form giving maximal rank. (For example, think of a 3×3 matrix of linear forms. For most choices of values to plug in for the variables, the determinant will be non-zero, so you get rank 3, but for special entries the determinant is 0 so the rank drops.) A linear form L for which the multiplication has maximal rank in all degrees is called a *Lefschetz element* for R/I.

Recall that the Hilbert function of an artinian graded algebra can be represented by a finite sequence of positive integers. If R/I is a graded artinian algebra then there is a last non-zero component, say $[R/I]_p$. Hence there is no hope that $\times L : [R/I]_t \to [R/I]_{t+1}$ is injective for all t, since in particular $[R/I]_p \to [R/I]_{p+1}$ is not injective. But there is hope that the WLP might hold. It is interesting to study what properties prevent WLP from holding and what properties guarantee it.

Lemma 8.4. *Let R/I be an artinian graded algebra and let L be a linear form. If $\times L : [R/I]_{t-1} \to [R/I]_t$ is surjective then $\times L : [R/I]_{t-1+r} \to [R/I]_{t+r}$ is surjective for all $r \geq 0$.*

Proof. Consider the exact sequence from Remark 3.12:

$$0 \to \left[\frac{I:L}{I}\right]_{t-1} \to \left[\frac{R}{I}\right]_{t-1} \xrightarrow{\times L} \left[\frac{R}{I}\right]_t \to \left[\frac{R}{\langle I, L\rangle}\right]_t \to 0.$$

In particular we have the exact sequence

$$[R/I]_{t-1} \xrightarrow{\times L} [R/I]_t \to [R/\langle I, L\rangle]_t \to 0$$

and the last vector space in this sequence is zero if and only if $\times L$ is surjective in that degree. But $R/\langle I, L\rangle$ is a standard graded algebra, so once it is zero in one degree, it is zero forever after. □

Exercise 48. Prove that if the Artinian algebra R/I has the WLP then the Hilbert function of R/I is unimodal. In fact, show that it is strictly increasing for a while, then non-increasing (but not necessarily strictly decreasing), but eventually zero. See the attached paper [HMNW] for a complete characterization of the shape of the Hilbert function of an algebra with the WLP (in fact the same description holds for SLP!). For example, the Hilbert function cannot be $(1, 4, 7, 6, 7, 3)$ even though one might hope that $\times L$ could be injective at first, then surjective, then injective again, then surjective.

Exercise 49. An important tool for studying Lefschetz properties for *monomial* algebras is the fact that R/I has the WLP (or SLP) if and only if the linear form given by the sum of the variables is a Lefschetz element. This was first proved in [MMN1], and we'll also talk about it in class. Write the proof carefully.

Exercise 50. Let $I = \langle x^2, y^2, z^2\rangle \subset R = k[x, y, z]$. For this exercise see also Examples 8.6 and 8.7.

(a) Prove that the Hilbert function of R/I is $(1, 3, 3, 1)$ (writing only the non-zero values).

(b) Let $L = x+y+z$. Show that $\times L$ is injective from degree 0 to degree 1 and surjective from degree 2 to degree 3.

(c) Show that $\times L$ is bijective from degree 1 to degree 2 if and only if $\text{char}(k) \neq 2$. Combining (b) and (c), conclude that R/I has the WLP if and only if $\text{char}(k) \neq 2$.

(d) If $\text{char}(k) = 2$, find an element in $[R/I]_1$ which is in the kernel of $\times(x + y + z)$ from degree 1 to degree 2.

Remark 8.5. 1. In the example given in Exercise 50, I is not saturated, but still it behaves in a much better way than the ideal in Example 3.5. This is because I is what is called a *complete intersection* (even though it is artinian as well).

2. Exercise 50 illustrates the fact that the characteristic sometimes plays an interesting role in the study of the Weak Lefschetz property, as do monomial ideals and as do complete intersections. Maybe one of the most important open questions about the WLP is whether all artinian complete intersections in ≥ 4 variables have the WLP, in characteristic zero. It is known to be true for *monomial* complete intersections, but not known for *arbitrary* complete intersections.

Exercise 51. This example appeared first in [BK] Example 3.1. Let $R = k[x, y, z]$ and $I = \langle x^3, y^3, z^3, xyz\rangle$.

(a) Prove that R/I is artinian.

(b) Find the Hilbert function of R/I.

(c) Show that R/I fails the WLP in any characteristic. (Hint: focus on the multiplication from degree 2 to degree 3. It is a fact, which you can use, that for studying WLP for a monomial ideal, it is enough to consider $\times L$ for $L = x + y + z$.)

The paper [MMN1] extends this, exploring the WLP more generally for monomial ideals in $n + 1$ variables having $n + 2$ minimal generators and containing powers of each of the variables (i.e. *almost complete intersections*). The ideal in Exercise 51 is a specific example of the case $n = 2$. The main results of [MMN1] are for $n = 2$. This paper is attached in the last part of these notes.

Example 8.6. Let $I = \langle x^2, y^3 \rangle \subset k[x, y]$, where k is any field. The Hilbert function of R/I is

$$\dim[R/I]_t = \begin{cases} 1 & \text{if } t = 0; \\ 2 & \text{if } t = 1; \\ 2 & \text{if } t = 2; \\ 1 & \text{if } t = 3; \\ 0 & \text{if } t \geq 4. \end{cases}$$

Since I is a monomial ideal, we can use Exercise 49 to study the WLP for R/I. So let $L = x + y$. We claim that $\times L : [R/I]_{t-1} \to [R/I]_t$ is

- injective for $t \leq 1$
- an isomorphism for $t = 2$
- surjective for $t \geq 2$.

Let us check what happens in the middle, i.e. from degree 1 to degree 2. Let $f = ax + by \in [R]_1 = [R/I]_1$. Then

$$Lf = (x + y)(ax + by) = (a + b)xy + by^2$$

(using the fact that $x^2 = 0$ in R/I). In order for Lf to be zero in R/I, then, we need $a = -b$ and $b = 0$. Thus $a = b = 0$ and so $\times L$ is an isomorphism in this degree as desired.

We will see later that by duality (see section 11 below), the calculation we just made in fact proves the full WLP in this example.

Example 8.7. Let $I = \langle x^3, y^3, z^3 \rangle \subset \mathbb{Z}_3[x, y, z]$. We leave it to you to check that the Hilbert function of R/I is the sequence $(1, 3, 6, 7, 6, 3, 1)$. Let $L = ax + by + cz$ be any linear form (even though we know that it is enough to study $L = x + y + z$). We claim that $\times L : [R/I]_2 \to [R/I]_3$ has a nonzero kernel, so R/I fails WLP. Indeed, working for now in R itself we have

$$L \cdot L^2 = L^3 = (ax + by + cz)^3 = a^3x^3 + b^3y^3 + c^3z^3 \in [I]_3$$

so $\times L$ does indeed have a nonzero kernel. This example stresses the important role that the field can play.

9. The Non-Lefschetz locus

The name "non-Lefschetz locus" was introduced in [BMMN1], and indeed the most thorough treatment can be found there. See also [Mar1], [Mar2] and [Mar3] for more recent work on this topic.

Definition 9.1. Let R/I be a standard graded k-algebra, where k is a field. The *non-Lefschetz locus* of R/I is the set $\mathcal{L}_{R/I}$ of linear forms of R that are not Lefschetz elements (i.e. such that the corresponding multiplication does not have maximal rank).

Remark 9.2.
1. Since multiplication by a nonzero scalar does not affect the rank of $\times L$, we view $\mathcal{L}_{R/I}$ as a subset of $\mathbb{P}^{n-1}$ rather than of $[R]_1$, where n is the number of variables.

2. $\mathcal{L}_{R/I}$ actually has a scheme structure, which we will not worry about here. But see [Mi1] and [BMMN1] (especially the latter) for details.
3. It is often convenient to restrict to the multiplication $\times L$ from a fixed degree to the next in R/I. In very nice situations (e.g. when R/I is Gorenstein), it is enough to find this locus in one specific degree in order to know it for all of R/I. Again see [BMMN1].
4. Notice that $\mathcal{L}_{R/I}$ could be empty and it could also be all of $\mathbb{P}^{n-1}$. In fact, by definition R/I fails WLP exactly when $\mathcal{L}_{R/I} = \mathbb{P}^{n-1}$.
5. In this section we have worked in the context of a graded k-algebra R/I. However, the definitions of WLP, of Lefschetz elements and of non-Lefschetz locus work for any graded R-module.

The notion of studying the linear forms that fail to give maximal rank for multiplication on graded modules is really a question about determinantal varieties, and as such is a classical idea. Next we give an example where we compute a non-Lefschetz locus, and we give an application to liaison theory due to Joe Harris.

Example 9.3. This example gives an interesting application of the non-Lefschetz locus to liaison theory, originally due to Joe Harris.

Let $I = \langle x, y, z, w^2 \rangle \subset \mathbb{C}[x, y, z, w]$. It's easy to check that

$$\dim[R/I]_t = \begin{cases} 1 & \text{if } t = 0, 1; \\ 0 & \text{if } t \neq 0, 1. \end{cases}$$

First let us find the non-Lefschetz locus for R/I. Let $L = ax + by + cz + dw \in [R]_1$. We want to know for which a, b, c, d is it true that $\times L$ fails to have maximal rank from degree 0 to degree 1. In this case, failure of maximal rank is equivalent to $\times L$ being the zero map.

Take as a basis for $[R/I]_0$ the element 1, and as a basis for $[R/I]_1$ the element w. Clearly $(ax+by+cz+dw)(1) = 0$ in R/I if and only if $d = 0$. Thinking of $[R]_1$ as an affine space, the non-Lefschetz locus is the hyperplane defined by $d = 0$. Projectivizing this, we get that the non-Lefschetz locus $\mathcal{L}_{R/I} \subset \mathbb{P}^3$ is the plane defined by $\mathbb{V}(d)$ (note that the variables defining this projective space are a, b, c, d).

This example originally arose in a very different setting, which we now describe (and was Harris' original motivation for his suggestion).

For curves in $\mathbb{P}^3$ (and in fact much more generally, but here we restrict the setting) there is an equivalence relation called *liaison*. Two curves are directly linked (essentially) if their union is a complete intersection. The notion of direct linkage *generates* an equivalence relation called *liaison*. (Direct linkage satisfies the symmetric property but not the reflexive or transitive properties.) There is a graded module called the *Hartshorne-Rao module*

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathcal{I}_C(t))$$

that is an invariant of the liaison class of C up to shifts and duals. Harris noticed that the non-Lefschetz locus (using the modern name) is an isomorphism invariant, so it has information for us about the liaison class. (See [Mi2] for details.)

Now let C be the disjoint union of a line λ and a conic Y in $\mathbb{P}^3$. λ meets the plane of Y in a point, P. It turns out that $M(C)$ is isomorphic to $R/(I_\lambda + I_Y)$. When $P = [0, 0, 0, 1]$, we get $M(C)$ is precisely the ring R/I of this example. Otherwise it differs by a change of variables. Omitting a lot of details, including a very powerful theorem of Rao from [Rao], one shows that if C' is another curve consisting of the disjoint union of a line and a conic then C is linked (in a finite number of steps) to C' if and only if C and C' share the same distinguished point P.

10. Hilbert functions of Gorenstein algebras

The study of Hilbert functions of artinian Gorenstein algebras is far from complete, but there are many fascinating results that are known. In this section we will describe some of this work, especially as it relates to the question of WLP and/or SLP.

We first remind the reader of the important fact that the Hilbert function of an artinian Gorenstein algebra is symmetric (see Remark 5.1). We will see that there is only one obvious condition on a symmetric Hilbert function that forces the WLP to hold, but there is much more in the direction of Hilbert functions that force WLP *not* to hold for Gorenstein algebras. And there is a lot that has been discovered in the non-WLP setting.

10.1. **Hilbert functions of Gorenstein algebras with the WLP.** As we have said, we have a complete understanding of the possible Hilbert functions of artinian Gorenstein algebras with the WLP. We will describe this in this subsection.

Definition 10.1. Let

$$\underline{h} = (1, h_1, h_2, h_3, \dots, h_{e-3}, h_{e-2}, h_{e-1}, h_e = 1)$$

be a symmetric vector of positive integers.

Consider the first difference sequence given by

$$g_i = h_i - h_{i-1} \ \text{ for } \ 1 \le i \le \left\lfloor \frac{e}{2} \right\rfloor$$

(see Remark 4.6). Then we say that $\underline{h}$ is an *SI-sequence* if both $\underline{h}$ and $\underline{g}$ are O-sequences (see Definition 4.10).

The term "SI-sequence" is named after Stanley and Iarrobino. The following exercise and theorem together give a complete classification of the Hilbert functions of artinian Gorenstein algebras with the WLP.

Exercise 52. Let R/I be an artinian graded Gorenstein algebra and let $\underline{h}$ be its Hilbert function. If R/I has the WLP then prove that $\underline{h}$ is an SI-sequence. [Hint: See Remark 3.12 and Proposition 11.1 below.]

We will see that the converse of the statement in Exercise 52 is not true (see the description of Ikeda's example below): if R/I is Gorenstein and the Hilbert function of R/I is an SI-sequence, it almost never forces R/I to have the WLP. See Remark 10.4, though. However, a partial converse does hold and it completes the classification of Hilbert functions of Gorenstein algebras with the WLP.

Theorem 10.2 ([Harima])**.** *If $\underline{h}$ is an SI-sequence (for any number of variables h_1) then there exists a standard graded artinian Gorenstein algebra R/I with Hilbert function $\underline{h}$, having the WLP.*

Now we briefly consider a special kind of Gorenstein algebra, and we will see that it is forced to have the WLP.

Definition 10.3. A *compressed* Gorenstein algebra is one for which the Hilbert function is as big as possible. Thanks to symmetry, this means that the Hilbert function is of the form

$$\left(1, 3, 6, \dots, \binom{d-1}{2}, \binom{d}{2}, \binom{d-1}{2}, \dots, 6, 3, 1\right)$$

in the case of even socle degree (i.e. the last non-zero entry is in even degree) and of the form

$$\left(1, 3, 6, \dots, \binom{d-1}{2}, \binom{d}{2}, \binom{d}{2}, \binom{d-1}{2}, \dots, 6, 3, 1\right)$$

in the case of odd socle degree.

Exercise 53. Verify that the Hilbert function of a compressed Gorenstein algebra is an SI-sequence.

Remark 10.4. In general, the Hilbert function of an algebra R/I (not necessarily Gorenstein) does not force it to have the WLP, nor to fail to have the WLP. However, there is a class of algebras for which the Hilbert function *does* force the WLP, and this was described in [MZ2]. We will omit details here.

For Gorenstein algebras of arbitrary codimension, though, there is one type of Hilbert function that clearly forces the WLP, and one trait of the Hilbert function that forces WLP to fail. We describe them now. (We do not in any way claim that either of these is the *only* example with the claimed property.)

First, if h is a compressed artinian Gorenstein algebra of even socle degree e and R/I has Hilbert function h then clearly R/I has the WLP. Indeed, up to and including degree $\frac{e}{2}$, I is zero so R/I coincides with R and multiplication by any linear form is injective. But we have reached the middle of the h-vector, so by duality all other maps are surjective, and WLP holds. Notice that if R/I has odd socle degree then even if it is compressed, the middle map can fail to be an isomorphism. This happens, for instance, in Ikeda's example [Ik] described below.

Second, we saw in Exercise 48 that if R/I (not necessarily Gorenstein) has the WLP then its Hilbert function is unimodal. Thus any artinian algebra whose Hilbert function is not unimodal must fail WLP.

But we now have an even stronger condition for Gorenstein algebras. In Exercise 52 we saw that an artinian Gorenstein algebra whose Hilbert function is not an SI sequence has no hope of having the WLP, even if it is unimodal. Artinian Gorenstein algebras whose Hilbert functions are unimodal but not SI have been studied in [MZ3]. This extends the observation in Remark 10.4 about non-unimodal Hilbert functions forcing the WLP to fail.

Exercise 54. Find a sequence that is

- symmetric;
- an O-sequence;
- unimodal;

but is not an SI-sequence. (You do not have to find an explicit algebra with these properties, only a sequence. But see [MZ3] for results on such algebras.)

We remark here that since your solution to this problem is at least an O-sequence, Macaulay's theorem (Theorem 4.10) guarantees that there is a standard graded artinian algebra with this Hilbert function; what's new here is that this algebra can never be Gorenstein.

10.2. **Hilbert functions of Gorenstein algebras not necessarily with the WLP.** If our artinian Gorenstein algebra R/I does not necessarily have the WLP, a great deal of very interesting research has been done to study the possible Hilbert functions, even though we are far from a complete classification as we had when WLP is assumed. In this subsection we will sketch some of what is known. We will divide our discussion according to the codimension.

10.2.1. *Two variables.* We will see shortly that in this setting *everything* has the SLP (at least in characteristic zero). However, for our purposes we recall from Exercise 38 that any artinian Gorenstein algebra over $k[x, y]$ is in fact a complete intersection. Thus the relevant fact is encapsulated by the following exercise.

Exercise 55. (a) Show that the Hilbert function of any complete intersection $k[x, y]/I$ has the form

$$(1, 2, 3, 4, \dots, m-1, m, m, \dots, m, m-1, \dots, 4, 3, 2, 1)$$

where the number of m's in the middle is arbitrary. With this notation, the ideal I is of the form $I = (f, g)$, where f has degree m and the degree of g is the degree where the second $m-1$ occurs in the Hilbert function. If f and g both have degree m then there is only one m in the Hilbert function.

(b) Confirm that such a sequence is an SI-sequence.

10.2.2. *Three variables.* We saw in the last section that in any number of variables, the SI-sequences are precisely the Hilbert functions of artinian Gorenstein algebras with the WLP.

On the other hand, we have noted that in three variables it is an open question whether all codimension 3 artinian Gorenstein algebras have the WLP. It may be surprising, then, to know the following fact, originally due to Richard Stanley (see also Zanello [Z]), that could be interpreted as suggesting that it might be true that all codimension 3 artinian Gorenstein algebras have the WLP. (But before you get too excited about this possiblity, see the situation in codimension 4.)

Theorem 10.5 ([St2])**.** *If R/I is a codimension 3 artinian Gorenstein algebra then it's Hilbert function is an SI-sequence.*

10.2.3. *Four variables.* The first thing to note is that in this situation it is known that WLP does not necessarily hold! (This explains why Question 11.4 below is only in codimension 3.) The first example is due to Ikeda ([Ik] Example 4.4). Her example has Hilbert function $(1, 4, 10, 10, 4, 1)$. Notice that this is unimodal and even compressed, so it gives the first example that although WLP implies unimodal (and in fact SI), the converse is not true.

In fact, there is suggestive evidence that the Hilbert function of a codimension 4 artinian Gorenstein algebra is always an SI sequence. Indeed, apart from the fact that there is no known counter-example, it was shown in [MNZ2] that any artinian Gorenstein algebra with Hilbert function

$$(1, 4, h_2, h_3, h_4, \dots, h_{e-3}, h_{e-2}, 4, 1)$$

and $h_4 \le 33$ has Hilbert function that is an SI sequence (which of course is even stronger than simply being unimodal). This was extended by Seo and Srinivasan [SeSr] to the case $h_4 = 34$.

So one can pose the question whether the Hilbert functions of all artinian Gorenstein algebras of codimension 4 are SI sequences, and we conjecture that the answer is "yes:"

Conjecture 10.6. *Let R/I be a codimension 4 artinian Gorenstein algebra with Hilbert function*

$$\underline{h} = (1, 4, h_2, h_3, h_4, \dots, h_4, h_3, h_2, 4, 1).$$

Then $\underline{h}$ is an SI-sequence.

If this turns out to be correct, it makes a very nice conjectural bridge from the codimension 3 case to the codimension 4 case to the codimension ≥ 5 case, since conjecturally in codimension 3 all Gorenstein algebras have the WLP and SI Hilbert functions, while in codimension 4 they definitely do not all have the WLP but nevertheless (conjecturally) all have SI Hilbert functions, and in codimension ≥ 5 we will see that the Hilbert functions are not even necessarily unimodal.

10.2.4. ≥ 5 *variables.* Recall from Exercise 48 that if the Hilbert function of an artinian Gorenstein algebra R/I is not unimodal (or even if it is unimodal but not SI) then R/I cannot have the WLP. Still, it is of great interest to try to understand these Hilbert functions for their own sake.

A lot of papers have been written on the general theme of "how non-unimodal can a Gorenstein sequence be?" Of course once it is non-unimodal then WLP does not hold, but still it is an interesting question to try to determine the extent to which non-unimodality is possible. So many papers have appeared on this topic that we will not make any effort here to try to list them all, and will just point to a few highlights.

The first non-unimodal Gorenstein sequence was found by Richard Stanley in 1978 [St2]. It is the sequence $(1, 13, 12, 13, 1)$. It was shown in [MZ1] that among Gorenstein algebras with socle degree 4 (meaning that the end of the Hilbert function is in degree 4), this has the smallest value of h_1, i.e. 13 is the smallest codimension that occurs among non-unimodal Gorenstein Hilbert functions of socle degree 4.

The first challenge, then, was to restrict to Gorenstein algebras of socle degree 4. Knowing that 13 is the smallest possible h_1, the natural question is to ask how big $h_1 - h_2$ can be. Not surprisingly, this depends on how big h_1 is. Quite a few papers have been written on this topic, but we will just mention that Stanley conjectured the following. For given value $h_1 = r$, let $f(r)$ be the smallest possible value for h_2. Then

$$\lim_{r \to \infty} \frac{f(r)}{r^{2/3}} = 6^{2/3}.$$

This conjecture was proven in [MNZ3]. Other asymptotic results (including for higher socle degree) have been proven (e.g. [MNZ4], [BGIZ]).

If one does not care about socle degree 4, it is known that non-unimodal Gorenstein examples exist for all codimensions ≥ 5. (Again, codimension 4 is open.) The first example in codimension 5 was given by D. Bernstein and A. Iarrobino in [BI]. In fact, it is known that a Gorenstein sequence can even have as many "valleys" as you like – this was shown by M. Boij [B]. Finally, we recall that in [MZ3] it was shown by J. Migliore and F. Zanello that artinian Gorenstein algebras exist whose Hilbert function is unimodal but is not SI, so also these algebras must fail WLP.

11. Proving WLP for artinian Gorenstein algebras, including complete intersections

Now let's return to the WLP question. We begin our discussion with some additional facts about artinian Gorenstein algebras, and connections between their Hilbert functions and the WLP question. Recall that we have already seen that if R/I is artinian Gorenstein then its Hilbert function is an SI-sequence, and all SI-sequences are represented by some artinian Gorenstein algebra, even if it is *not* true that SI alone implies that R/I has the WLP (as evidenced by Ikeda's example [Ik]).

The reader may have noticed that the definition of WLP, and of non-Lefschetz locus, involves a consideration of all of the maps between components of R/I, or of M in the more general setting of graded modules – see Remark 9.2 (5). In the case of graded modules, we have no choice (in general) but to look at all pairs of consecutive components. However, for k-algebras R/I it often happens that we can prove shortcuts and work around this issue. The first instance of this is Lemma 8.4, where we saw that for any artinian R/I (not necessarily Gorenstein), once $\times L$ is surjective in one spot, it is automatically surjective from that point on.

The best of all worlds is the case of Gorenstein algebras (including complete intersections). The important starting point to studying WLP for artinian Gorenstein algebras is the following. Recall that for an artinian algebra R/I, the *socle degree* is the degree of the last non-zero component of R/I. Also, for a real number t, $\lceil t \rceil$ is the "round-up" of t (e.g. $\lceil \frac{5}{3} \rceil = 2$), and analogously for the round-down $\lfloor t \rfloor$ (e.g. $\lfloor \frac{5}{3} \rfloor = 1$).

Proposition 11.1. *Let R/I be Gorenstein of socle degree e and let L be a general linear form. The following are equivalent.*

1. *R/I has WLP.*
2. *$\times L : [R/I]_{t-1} \to [R/I]_t$ is injective for all $t \le \lceil \frac{e}{2} \rceil$.*
3. *$\times L : [R/I]_{t-1} \to [R/I]_t$ is surjective for all $t \ge \lceil \frac{e+1}{2} \rceil$.*
4. *$\times L : [R/I]_{\lceil \frac{e}{2} \rceil - 1} \to [R/I]_{\lceil \frac{e}{2} \rceil}$ is injective.*
5. *$\times L : [R/I]_{\lceil \frac{e+1}{2} \rceil - 1} \to [R/I]_{\lceil \frac{e+1}{2} \rceil}$ is surjective*

Remark 11.2. The point of this proposition is to realize that for artinian Gorenstein algebras, injectivity on the left half is equivalent to surjectivity on the right half, and furthermore there is one spot whose injectivity implies the full WLP, and one spot where the surjectivity implies the full WLP. Furthermore, when e is odd, the spots coincide and we can look for either injectivity or surjectivity, whichever may be easier.

Proof of Proposition 11.1. The heart of the matter is Remark 5.1. In general, when R/I is artinian, its k-dual is isomorphic to a twist of the canonical module, so when R/I is Gorenstein as well, up to twist R/I is self-dual.

Since R/I has socle degree e, by self-duality, in particular we have $\dim_k [R/I]_e = 1$. Then if we make a suitable choice of bases for all the homogeneous components of R/I, a matrix representing $\times L$ from degree $t-1$ to degree t is the transpose of the matrix for $\times L$ from degree $e-t$ to degree $e-t+1$.

The numerical conditions in items 2. and 3. represent the degrees "closest to the middle" where we expect injectivity (respectively surjectivity). For example, for the Hilbert function $(1,3,3,1)$ we have $e = 3$ and both $\lceil \frac{e}{2} \rceil$ and $\lceil \frac{e+1}{2} \rceil$ represent $t = 2$, so both refer to the map from degree 1 to degree 2. On the other hand, when the Hilbert function is $(1,3,6,3,1)$ we have $e = 4$, so the bound $\lceil \frac{e}{2} \rceil$ in condition 2. represents the map from degree 1 to degree 2 (the last place where we expect injectivity, while the bound $\lceil \frac{e+1}{2} \rceil$ in condition 3. represents the map from degree 2 to degree 3 (the first place where we expect surjectivity).

It's clear that condition 1. implies all of the other conditions, and that 2. and 3. together imply 1. It's also clear that condition 2. implies condition 4. and condition 3. implies condition 5. The fact that 2. and 3. are equivalent comes from the self-duality and the above observation about the matrices. (The rank of a matrix is unaffected by taking the transpose.)

The fact that 4. and 5. are equivalent also comes from self-duality, noticing that in one situation (e odd) we are talking about the same map and getting that it is an isomorphism (injectivity is equivalent to surjectivity).

The implication 5. implies 3. comes from Lemma 8.4. This completes the proof. (Note that we never directly prove that 4. implies 2., but rather invoke self-duality to get it for free.) □

Two of the most important open questions about the WLP are the following.

Question 11.3. *In characteristic zero, does every artinian complete intersection, in any number of variables, have the WLP? Same question for SLP.*

Question 11.4. *In characteristic zero, does every artinian Gorenstein algebra in three variables have the WLP? Same question for SLP.*

In the case of Question 11.3, the conjecture that the answer is "yes" first appeared in [RRR]). In the case of Question 11.4, it was mentioned in [Ik] that "(i)t is conjectured" to be true in codimension 3, without an explicit reference; it was also explicitly conjectured in [BMMNZ2] in codimension 3. The most complete results in codimension 3 about Question 11.4 can be found in the latter paper. We will describe the situation more carefully below. But we begin with the complete intersection situation.

11.1. **The WLP for complete intersections.** We will first focus on Question 11.3. Our main goal in this subsection is to describe what is known about this question.

Let $R = k[x_1, \dots, x_n]$, where (as usual) k has characteristic zero, and let $I = \langle F_1, \dots, F_n \rangle$ be a complete intersection. Let $d_i = \deg F_i$ for $1 \le i \le n$. We start with a by-now classical result for a special choice of the F_i. In our opinion, this theorem launched the entire field of Lefschetz theory that this school is about, since it leads to questions about complete intersections, Gorenstein algebras, monomial ideals, level algebras, and more.

Theorem 11.5 ([St1], [W], [RRR])**.** *Let $I = \langle x_1^{d_1}, \dots, x_n^{d_n} \rangle$. Then R/I has the SLP.*

For the next result, note that the space parametrizing complete intersections of fixed generator degrees is irreducible, so a "general complete intersection" makes sense. (It is understood in using the term "general" that the generator degrees are fixed.)

Corollary 11.6. *A general complete intersection in any number of variables has the SLP.*

The idea is that since the parameter space is irreducible, by semicontinuity it is enough to find one example where SLP holds in order to say that it holds for the general complete intersection, and the prior result provides that example.

Beyond this result, what we know is quite sparse. It is convenient to describe the results according to the number of variables. We remind the reader that we are assuming characteristic zero, and we will not keep restating that. Also, following convention, by "artinian Gorenstein algebra R/I of codimension n" we mean that the polynomial ring R has n variables.

11.1.1. *Two variables.* In this case everything has SLP:

Theorem 11.7 ([HMNW] Proposition 4.4 attached to these notes)**.** *If $n = 2$ then for any homogeneous ideal J, R/J has the SLP. In particular, of course, all complete intersections have the WLP.*

11.1.2. *Three variables.*

Theorem 11.8 ([HMNW] Theorem 2.3 and Corollary 2.4 attached to these notes)**.** *If $n = 3$ then every complete intersection has the WLP.*

Proof. Here is the idea of the proof from [HMNW] (see the attached paper). Let $I = (F_1, F_2, F_3)$ be a complete intersection, and assume that $d_i = \deg F_i$. For convenience assume $d_1 \leq d_2 \leq d_3$.

When $d_3 \geq d_1 + d_2 - 3$, a simpler proof was already known from work of Watanabe [W2] that R/I has the WLP. So we can assume without loss of generality that $d_3 < d_1 + d_2 - 3$.

Start with the minimal free resolution of the complete intersection ideal $I = (F_1, F_2, F_3)$. Then we have the Koszul resolution

$$0 \to R(-d_1 - d_2 - d_3) \to \bigoplus_{1 \leq i < j \leq 3} R(-d_i - d_j) \xrightarrow{\phi} \bigoplus_{i=1^3} R(-d_i) \xrightarrow{[F_1,F_2,F_3]} R \to R/I \to 0.$$

Consider the commutative diagram of graded modules obtained from the Koszul resolution and considering multiplication by a general linear form L:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & E(-1) & \to & \mathbb{F}_1(-1) & \xrightarrow{[F_1,F_2,F_3]} & R(-1) & \to & (R/I)(-1) & \to & 0 \\
 & & & & \downarrow M & & \downarrow \times L & & \downarrow \times L & & \\
0 & \to & E & \to & \mathbb{F}_1 & \xrightarrow{[F_1,F_2,F_3]} & R & \to & R/I & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \bar{F}_1 & & \bar{R} & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

where:

- $\mathbb{F}_1 = \bigoplus_{i=1}^3 R(-d_i)$;
- E is the kernel of the homomorphism given by $[F_1, F_2, F_3]$. (This is the syzygy module – it is also the image of ϕ in the Koszul resolution above. The cokernel of ϕ is the ideal I.)
- the bars $\bar{F}_1$ and $\bar{R}$ denote the restriction of these free modules to $R/(L) \cong k[x, y]$;

- M is the matrix $\begin{bmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & L \end{bmatrix}$;

Notice that the first vertical exact sequence in the commutative diagram is the direct sum of three copies of the exact sequence

$$0 \to R(-1) \xrightarrow{\times L} R \to \bar{R} \to 0$$

twisted by $-d_1, -d_2, -d_3$ respectively.

We then sheafify. It turns out that the sheafification of E is a locally free sheaf (because R/I is artinian). Let λ be the line in $\mathbb{P}^2$ defined by L. We get the commutative diagram of sheaves

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E}(-1) & \to & \mathcal{F}_1(-1) & \xrightarrow{[F_1,F_2,F_3]} & \mathcal{O}_{\mathbb{P}^2}(-1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E} & \to & \mathcal{F}_1 & \xrightarrow{[F_1,F_2,F_3]} & \mathcal{O}_{\mathbb{P}^2} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E}|_\lambda & \to & \overline{\mathcal{F}_1} & \xrightarrow{[\bar{F}_1,\bar{F}_2,\bar{F}_3]} & \mathcal{O}_\lambda & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

Notice that $\bigoplus_{t\in\mathbb{Z}} H^1(\mathcal{E}(t)) \cong R/I$.

Now the whole proof hinges on applying the Grauert-Mülich theorem to $\mathcal{E}$. (This is a theorem that talks about the splitting type of the restriction of a vector bundle to a general line.) Our assumption that $d_3 < d_1 + d_2 - 3$ forces $\mathcal{E}$ to be semistable, which means that we can apply Grauert-Mülich.

Consider the restriction $\mathcal{E}|_\lambda$. A theorem of Grothendieck says that this restriction splits as a direct sum $\mathcal{O}_\lambda(a) \oplus \mathcal{O}_\lambda(b)$. Grauert-Mülich then says that $|a-b| \leq 1$. Using the commutative diagram of sheaves above, cohomology, and the Snake Lemma, we get (after some details for which we refer you to the attached paper) that

$$\times L : [R/I]_{t-1} \to [R/I]_t$$

has to be surjective, because for each t either $h^0(\mathcal{E}|\lambda(t) = 0$ or $h^1(\mathcal{E}|\lambda(t) = 0$. □

Unfortunately, this method does not extend very much. Indeed, very little is known about SLP for codimension 3 complete intersections, although Marangone [Mar3] has some results for multiplication by forms of degree 2. Similarly, not so much is known about WLP in more variables, as we will see now.

11.1.3. *Four variables.* In four variables even less is known. As before, we start with complete intersections.

Theorem 11.9 ([BMMN2] Proposition 7.5)**.** *Let $A = R/I$ where $I = \langle F_1, F_2, F_3, F_4\rangle$ and $\deg F_i = d_i$. Set $d_1 + d_2 + d_3 + d_4 = 3\lambda + r$, $0 \leq r \leq 2$. Let L be a general linear form. Then the multiplication maps $\times L : [A]_{t-1} \to [A]_t$ are injective for all $t < \lambda$.*

Now we specialize to the equigenerated case, i.e. we assume that $d_1 = \cdots = d_4 = d$ for some positive integer d. We'll start with the codimension 4 version of a result of Alzati and Re (proved earlier by Ilardi in the special case of Jacobian ideals) – note that there is a more general version of their theorem that we will mention in the next subsection.

Theorem 11.10 ([AR] Corollary 4)**.** *Let $A = R/I = R/\langle F_1, F_2, F_3, F_4\rangle$, where $\deg F_i = d$ for all i. Let L be a general linear form. Then $\times L : [A]_{t-1} \to [A]_t$ is injective for all $t \le d$.*

Improving this we have a simple corollary of Theorem 11.9:

Corollary 11.11. *Let $A = R/I$ where $I = \langle F_1, F_2, F_3, F_4\rangle$ and $\deg F_i = d$ for some integer d. Let L be a general linear form. Then the multiplication maps $\times L : [A]_{t-1} \to [A]_t$ are injective for all $t < \frac{4d-2}{3}$.*

And improving this even further we have another result from [BMMN2] that assumes right from the beginning that the ideal is equigenerated, but as a result gives a stronger conclusion.

Theorem 11.12 ([BMMN2] Theorem 4.9)**.** *Let $A = R/I = R/\langle F_1, F_2, F_3, F_4\rangle$, where $\deg F_i = d$ for all i. Let L be a general linear form. Then $\times L : [A]_{t-1} \to [A]_t$ is injective for all $t < \lfloor \frac{3d+1}{2} \rfloor$.*

The proofs of Theorem 11.9 and Theorem 11.12 are completely different. The first uses an analysis of rank three vector bundles, while the second studies the geometry of a certain union of two smooth complete intersection curves in $\mathbb{P}^3$.

Remark 11.13. We recall (Proposition 11.1) that to prove WLP it is enough to prove injectivity for $t = 2d-2$, so Theorem 11.12 covers roughly half the distance between Theorem 11.10 and the optimal result that is still open.

11.1.4. *Five or more variables.* Naturally even less is known in the case of five or more variables. We remind the reader of Theorem 11.5 and its corollary for general complete intersections of fixed generator degree.

One result that we do have is the full version of Theorem 11.10:

Theorem 11.14 ([AR] Corollary 4)**.** *Let $A = R/I = R/\langle F_1, \dots, F_n\rangle$, where $\deg F_i = d$ for all i. Let L be a general linear form. Then $\times L : [A]_{t-1} \to [A]_t$ is injective for all $t \le d$.*

As with the case of four variables, this result was also shown by Ilardi in the special case where I is a Jacobian ideal. In particular, Alzati and Re proved:

Corollary 11.15. *When $n = 5$, a complete intersection of quadrics has the WLP.*

11.2. **The WLP for codimension 3 artinian Gorenstein algebras.** As we have seen, it is known that in codimension 2 all artinian algebras (not only Gorenstein) have the WLP (even the SLP), while in codimension ≥ 4 there exist artinian Gorenstein algebras failing the WLP. However, note again that the full WLP for Gorenstein algebras with $n = 3$ is still open. Thus this case merits its own subsection.

Remark 11.16. Let us repeat an observation made before. We defined SI sequences above in arbitrary codimension, in Remark 10.1. We saw that SI sequences are exactly the possible Hilbert functions of artinian Gorenstein algebras with WLP, in any codimension. On the other hand, without invoking WLP, it is known that in codimension 3 the SI sequences are exactly the Hilbert functions of artinian Gorenstein algebras [St2], [Z]. These two facts strongly suggest that all codimension 3 Gorenstein algebras will have WLP, but the question is still open. Furthermore, we saw that the case of codimension 4 provides a cautionary note because conjecturally all such Hilbert functions are SI-sequences, but we know that not all such algebras have the WLP.

The paper [BMMNZ2] reduced the WLP problem to one involving compress artinian Gorenstein algebras:

Theorem 11.17 ([BMMNZ2] Corollary 2.5). *If all codimension 3 artinian compressed algebras of odd socle degree have the WLP then all codimension 3 artinian Gorenstein algebras have the WLP.*

At first sight this seems to make the job much easier, since rather than study *all* codimension 3 artinian Gorenstein algebras, it is enough to consider only the compressed ones. However, in the same paper [BMMNZ2], a great deal of work (involving some very pretty geometry!) went into proving just the case $(1,3,6,6,3,1)$. (I've always been intrigued by this problem, and maybe for this reason, when my car reached 136,631 miles in 2011, I stopped the car to take a picture of the odometer:

Luckily I was not driving on the highway at the time, as you can see from the speedometer!) More generally, [BMMNZ2] showed the following, which removes the assumption on the characteristic.

Theorem 11.18 ([BMMNZ2] Theorem 3.8). *Any artinian Gorenstein algebra R/I with Hilbert function $(1,3,6,6,3,1)$ has the WLP, unless the characteristic of k is 3 and the ideal is $I = (x^2y, x^2z, y^3, z^3, x^4 + y^2z^2)$ after a change of variables.*

Putting several things together (and avoiding details here) the same paper showed the following:

Corollary 11.19 ([BMMNZ2] Corollary 3.12 and Corollary 3.13). *Assume characteristic zero. Then*

1. *All codimension 3 artinian Gorenstein algebras of socle degree at most 6 have the WLP.*
2. *All codimension 3 artinian Gorenstein algebras of socle degree at most 5 have the SLP.*

12. Beyond the WLP in unexpected directions

12.1. **Vanishing conditions on a linear system.** Let P be a point in $\mathbb{P}^n$ and m a positive integer. A *point of multiplicity* m *supported at* P, denoted by mP, is the geometrical object defined by the ideal $I_{mP} = (I_P)^m$. In particular for $m = 1$ the point P is said to be *reduced*.

More generally, given a set of distinct points $X = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ and positive integers $m_1, \dots, m_s$, the set of points supported at X with multiplicity $m_1, \dots, m_s$ is the union of the points, denoted by $Z = m_1P_1 + \cdots + m_sP_s$, that is defined by the ideal

$$I_Z = (I_{P_1})^{m_1} \cap \cdots \cap (I_{P_s})^{m_s}.$$

We say that mP, a point of multiplicity m, imposes r *independent conditions* on the forms of degree t of an ideal $I \subseteq \mathbb{C}[\mathbb{P}^n]$ if

$$\dim_{\mathbb{C}}[I \cap (I_P)^m]_t = \dim_{\mathbb{C}}[I]_t - r.$$

More generally, we say that a subscheme $Z \subset \mathbb{P}^n$ imposes r *independent conditions* on the forms of degree t of an ideal $I \subset \mathbb{C}[\mathbb{P}^n]$ if

$$\dim_{\mathbb{C}}[I \cap I_Z]_t = \dim_{\mathbb{C}}[I]_t - r.$$

We will primarily be interested in the case when Z is a finite set of points and when $Z = mP$ for a point of multiplicity m. When Z is a finite set of points and $r = |Z|$, we sometimes simply say that Z *imposes independent conditions on* $[I]_t$. In Example 7.3 we used this idea to compute the number of independent conditions imposed by a line $C \subseteq \mathbb{P}^3$ on the forms of degree t.

Exercise 56. Prove that in order to show that a finite set of points Z imposes independent conditions on $[R]_t$, it is enough to show that for *each* $P \in Z$ there is a form of degree t vanishing on $Z \backslash \{P\}$ but not vanishing at P.

Exercise 57. Let $P \in \mathbb{P}^n$ be a point. Compute the number of independent conditions that mP imposes on forms of degree t in $\mathbb{C}[x_0, x_1, \dots, x_n]$. In particular show that this number is at most $\binom{m+n-1}{n}$. Hint: it is not restrictive to take $P = [1, 0, \dots, 0]$.

The binomial coefficient $\binom{n+m-1}{n}$, calculated in Exercise 57, represents the maximum number of independent conditions that a point P of multiplicity m can impose on any linear system of forms of degree d. Sometimes, as the same example shows, this number of independent conditions cannot be achieved just for numerical reasons. This happens when the dimension of the linear system is not large enough.

The next two exercises show that different points might impose a different number of conditions on a linear system.

Exercise 58. Let X be the following set of 8 points in $\mathbb{P}^2$

$$X = \left\{ \begin{array}{ccc} [-1,1,1] & [0,1,1] & \\ [1,0,1] & [0,0,1] & [-1,0,1] \\ [1,-1,1] & [0,-1,1] & [-1,-1,1] \end{array} \right\}$$

(a) In the affine space given by $x_2 \neq 0$, these points correspond to

$$\{(-1, 1), (0, 1), (1, 0), (0, 0), (-1, 0), (1, -1), (0, -1), (-1, -1)\}.$$

Sketch this set of points, noting the collinearities.

(b) Compute the Hilbert function of X.

Exercise 59. Let X be the following set of 8 points in $\mathbb{P}^2$

$$X = \left\{ \begin{array}{ccc} [-1,1,1] & [0,1,1] & \\ [1,0,1] & [0,0,1] & [-1,0,1] \\ [1,-1,1] & [0,-1,1] & [1,-1,1] \end{array} \right\}$$

Compute the number of conditions imposed by $P = [1,1,1]$ on $[I_X]_3$. How many conditions does the point $P' = [1,0,0]$ impose on $[I_X]_3$?

Remark 12.1. From Exercise 59, in particular we have that both X and $X \cup \{P\}$ impose the same number (8) of independent conditions on forms of degree 3 in $\mathbb{C}[\mathbb{P}^2]$. It is a special case of the so called Cayley-Bacharach Theorem.

Theorem 12.2 (Cayley-Bacharach Theorem)**.** *Let C and C' be two cubic curves in $\mathbb{P}^2$ such that $X = C \cap C'$ is a set of nine distinct points. Let $Y \subseteq X$ be a set of eight points. Then any cubic curve vanishing at Y also vanishes at X. That is, any cubic through eight of the nine points must vanish also at the ninth point.*

The core of the proof is to show that we always have $H_Y = (1,3,6,8,\ldots)$. Indeed, in such case we easily have $[I_X]_3 = [I_Y]_3$.

12.2. **Unexpected curves and hypersurfaces.** Exercise 59 underscores that, given a set of points X, special points can fail to impose a condition on forms of a certain degree vanishing at X. However, if $[I_X]_t \neq (0)$ then a *general point* always imposes a condition on $[I_X]_t$. (The latter sentence means that the set of points that do *not* impose a condition on $[I_X]_t$ is a proper closed set of $\mathbb{P}^n$. Of course any point of X lies in this closed set.)

Exercise 60. Let X be a set of points in $\mathbb{P}^n$. Let t be such that $\dim_{\mathbb{C}}[I_X]_t > 0$. Show that there exists a point $P \in \mathbb{P}^n$ such that P imposes a condition on $[I_X]_t$.

Thus, it is natural to ask how many conditions a general point P of multiplicity m imposes on $[I_X]_t \neq (0)$. Recall that, from Exercise 57, the maximum number of conditions imposed by mP on $[I_X]_t$ is $\binom{m+n-1}{n}$.

Given a set of points $X \subseteq \mathbb{P}^n$ and two positive integers d, m, the *virtual dimension* of the linear system of the forms of degree d vanishing at X and at a general point of multiplicity m is

$$\text{v-dim}(X,d,m) = \dim_{\mathbb{C}}[I_X]_d - \binom{m+n-1}{n}.$$

Hence, the virtual dimension could be a negative integer for small values of d, and in these cases it certainly does not represent the dimension of any linear system. To avoid this issue we introduce the *expected dimension* of the linear system of the forms of degree d vanishing at X and at a general point of multiplicity m; it is

$$\text{e-dim}(X,d,m) = \max\{0, \text{v-dim}(X,d,m)\}.$$

Finally we have the *actual dimension* of the linear system of the forms of degree d vanishing at X and at a general point of multiplicity m, that is

$$\text{a-dim}(X,d,m) = \dim_{\mathbb{C}}[I_X \cap I_P^m]_d.$$

The numbers e-dim(X,d,m) and a-dim(X,d,m) are equal when mP imposes the maximum number of possible conditions. In general, from the definition we have a-dim$(X,d,m) \geq$

e-dim(X, d, m). However, the actual dimension is not necessarily equal to the expected dimension. Examples in $\mathbb{P}^2$ are easier using non-reduced points.

Exercise 61. Let $P_1 = [0,0,1], P_2 = [0,1,0], P_3 = [1,0,0] \in \mathbb{P}^2$. Consider the set $X = 2P_1 + P_2 + P_3$ and let P be a general point. Compute e-dim$(X, 4, 4)$ and a-dim$(X, 4, 4)$.

Remark 12.3. It is not possible to reproduce in $\mathbb{P}^2$ the situation in Exercise 61 by using sets of reduced points. Indeed, Any set of reduced points X in $\mathbb{P}^2$ has a-dim$(X, d, d) =$ e-dim(X, d, d). This is a consequence of Bezout's Theorem. Indeed, a curve of degree d vanishing at X and at a general point P with multiplicity d must contain as a component the union of the lines spanned by P and each of the points in X.

Then, if $d \leq |X| - 1$ we have a-dim$(X, d, d) = 0$; otherwise

$$\text{a-dim}(X, d, d) = \dim[I_P^d]_d - |X| = d + 1 - |X|$$

and

$$\text{e-dim}(X) = \dim[I_X]_d - \binom{d+1}{2} = \binom{d+2}{2} - |X| - \binom{d+1}{2} = d + 1 - |X|.$$

Definition 12.4. Let $X \subseteq \mathbb{P}^n$ be a (reduced) finite set of points. We say that X admits an *unexpected hypersurface* (*unexpected curve* if $n = 2$) of degree d with a general point P of multiplicity m if

$$\text{a-dim}(X, d, m) > \text{e-dim}(X, d, m).$$

The study of linear systems not having expected dimension is a classical topic in mathematics. However, the problem of determining unexpected curves and hypersurfaces as in the terms of Definition 12.4 was introduced in [CHMN] and [HMNT] and opened a new area of research; see [HMN] for a recent survey on the state of the art.

It is clear that for any finite set X, if $d < m$ then a-dim$(X, d, m) = 0$, hence X admits no unexpected hypersurfaces with respect these parameters. An interesting instance of Definition 12.4 is the case $d = m$. A reduced hypersurface of degree d with a point of multiplicity d must be a cone with vertex at that point, so in this case we say that X *admits an unexpected cone* of multiplicity d with vertex at a general point.

Remark 12.3 shows that no sets of reduced points admit unexpected cones on $\mathbb{P}^2$; however examples exist in higher dimensional spaces.

Example 12.5. Let $R = \mathbb{C}[x, y, z, w]$ Consider the following set of 9 points in $\mathbb{P}^3$.

$$X = \begin{array}{ccc} [1,0,0,0] & [0,1,0,0] & [1,1,0,0], \\ [0,0,1,0] & [0,0,0,1] & [0,0,1,1], \\ [1,0,1,0] & [0,1,0,1] & [1,1,1,1]. \end{array}$$

Such a set X is called a $(3,3)$-grid; it is the intersection of 3 lines in one ruling of the smooth quadric surface defined by the form $xw - yz$, with a set of 3 lines in the other ruling. These lines are defined by $\mathcal{L} = \{(z, w), (x, y), (x - z, y - w)\}$ and $\mathcal{H} = \{(y, w), (x, z)(x - y, z - w)\}$.

The Hilbert function of X is

$$H_X = (1, 4, 9, 9, \ldots).$$

Then, $\dim[I_X]_3 = 20 - 9 = 11$ and a general point of multiplicity 3 imposes on $[I_X]_3$ at most $\binom{5}{2} = 10$ independent conditions. Hence

$$\text{e-dim}(X, 3, 3) = 1.$$

However, if P is a general point, the surfaces consisting of the union of the planes spanned by the lines in $\mathcal{L}$ and P, and the union of the planes spanned by the lines of $\mathcal{M}$ with P, give two different cones of multiplicity 3 with vertex at the general point, hence

$$\text{a-dim}(X, 3, 3) \geq 2.$$

Thus, X admits an unexpected cone of degree 3.

12.3. **Geproci sets.** The set X in Example 12.5 is called a grid. We give below the general definition.

Definition 12.6. For a, b non negative integers, a set of ab points $X \subseteq \mathbb{P}^3$ is called an (a, b)-*grid* if there are two sets $\mathcal{L} = \{\ell_1, \ldots, \ell_a\}$ and $\mathcal{L}' = \{\ell'_1, \ldots, \ell'_b\}$, each containing pairwise skew lines, such that X is the set of the intersection points of the curves $\cup\mathcal{L}$ and $\cup\mathcal{L}'$. For grids we usually adopt the convention that $a \leq b$.

Exercise 62. Show that if $a \leq 2$ and $b \geq 4$ then an (a, b)-grid necessarily lies on a smooth quadric surface, but the defining grid lines do not. On the other hand, for $a \geq 3$ it does. *Hint: you can use the fact that a set of three skew lines in* $\mathbb{P}^3$ *lies on a unique smooth quadric surface.*

The relation between grids and unexpected cones is studied in detail in [CM2]. In particular it is shown in [CM2, Theorem 3.5.] that any (a, b)-grid X with $b \geq a \geq 2$ and $b \geq 3$ has an unexpected cone of degree a. Furthermore, if $a, b \geq 3$ then X also has an unexpected cone of degree b.

Remark 12.7. An interesting fact about grids is their particular behaviour under general projections. If X is an (a, b)-grid then the ab points of X lie on two space curves, namely $\gamma = \ell_1 \cup \ldots \cup \ell_a$ and $\gamma' = \ell'_1 \cup \ldots \cup \ell'_b$ which have no common components. Considering a general point P and a plane $H \cong \mathbb{P}^2$, we note that $\pi_P(X)$, the projection of X from P to H, is a complete intersection in H of type (a, b). Indeed, since P is general, $\pi_P(\gamma)$ and $\pi_P(\gamma')$ are two curves of degree a and b meeting transversally in $\pi_P(X)$.

The above property is formalized in the next definition.

Definition 12.8. Let X be a finite set of points in $\mathbb{P}^n$. We say that X is a *geproci set* if the general projection of X to $\mathbb{P}^{n-1}$ is a complete intersection.

It is clear that when X is a degenerate complete intersection in $\mathbb{P}^n$ then X is trivially a geproci set. A systematic study of geproci sets can be found in [POLITUS]. In particular, no example of a non-degenerate geproci set is known in $\mathbb{P}^n$ for $n \geq 4$. So, it makes sense to restate and refine the definition of geproci sets for the 3-dimensional case.

Definition 12.9. Let X be a finite set of points in $\mathbb{P}^3$. We say that X is an (a, b)-geproci set if the general projection of X to $\mathbb{P}^2$ is a complete intersection of two curves of degree a and b. Again, we use the convention $a \leq b$.

Remark 12.10. Remark 12.7 says that if X is a grid then it is the intersection in $\mathbb{P}^3$ of a curve of degree a and a curve of degree b, which immediately explains why it is geproci. Another interesting fact [CM] is that this is the only possible example of a curve of degree a and a curve of degree b in $\mathbb{P}^3$ meeting in ab points and having non-degenerate union (as long as $2 \leq a \leq b$). So non-grid geproci sets are much more subtle to study: the general projection $\pi(X)$ is the intersection of a curve of degree a and a curve of degree b, but X

itself is not. This is sharpened in [POLITUS2], where it is shown that if X is (a,b)-geproci and lies on a curve of degree a or a curve of degree b then it must actually be a grid.

Remark 12.11. The semicontinuity theorem ensures that if X is an (a,b)-geproci set, then the projection from every point (not necessarily general) in $\mathbb{P}^3$ of X is contained in a curve of degree a.

Any (a,b)-grid is an (a,b)-geproci set. It was shown in [CM, Theorem 5.12.] that the only non degenerate $(3,3)$-geproci sets are $(3,3)$-grids. (The same is true for nondegenerate $(2,b)$-geproci sets.)

Exercise 63. Let X be a set of six points in linear general position (no three points on a line and no four on a plane). Prove that X is not a $(2,3)$-geproci set. *Hint: Use Semicontinuity theorem and project from a special point.*

Exercise 64. Let X be a non degenerate $(2,b)$-geproci set, $b \geq 3$. Show that X is a $(2,b)$-grid. *Hint: Use Exercise 63.*

The first non degenerate and non-grid example is a $(3,4)$-geproci set that is the projectivization of the root system D_4 (see [HMNT]); we illustrate it in the next example.

Example 12.12. Let

$$X_{D_4}: \begin{array}{ccc} [1,1,0,0] & [1,0,1,0] & [0,1,-1,0] \\ [0,1,1,0] & [0,0,1,1] & [0,1,0,-1] \\ [1,0,-1,0] & [1,0,0,-1] & [0,0,1,-1] \\ [1,-1,0,0] & [1,0,0,1] & [0,1,0,1] \end{array}$$

Denote by P_{ij} the elements in the first three rows in the above array and by Q_1, Q_2, Q_3 the points in the last row. Let π be a general projection to a hyperplane. In order to show that X_{D_4} is a $(3,4)$-geproci set we need to prove that $\pi(X_{D_4})$ is complete intersection of a cubic curve and a quartic curve. However, note that the points in each row in the above table are collinear and so are their general projections. Thus a quartic curve containing all the points of the configuration is the union of the projection of these lines.

We note that the following sets

$$G_1 = \left\{ \begin{array}{ccc} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{array} \right\} \quad G_2 = \left\{ \begin{array}{ccc} Q_1 & P_{31} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{12} & Q_2 & P_{33} \end{array} \right\} \quad G_3 = \left\{ \begin{array}{ccc} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{33} & Q_3 \\ P_{31} & Q_2 & P_{22} \end{array} \right\}$$

are grids (indeed, observe that X_{D_4} is closed under the involution maps $\varphi([x,y,z,w]) = [-x,y,z,w]$ and $\psi([x,y,z,w]) = [x,y,z,-w]$ and we have $G_2 = \varphi(G_1)$, $G_3 = \psi(G_1)$).

Hence $\pi(G_1), \pi(G_2), \pi(G_3)$ define pencils of cubic curves in $\mathbb{P}^2$. Moreover, by the Cayley-Bacharach Theorem, any eight points of G_1 are enough to define the same pencil of cubics as all of G_1 does.

We claim that $\pi(G_1) \cup \{\pi(Q_1)\}, \pi(G_2) \cup \{\pi(P_{11})\}, \pi(G_3) \cup \{\pi(P_{23})\}$ determine the same cubic curve, which vanishes in all the twelve points of $\pi(X_{D_4})$. The set $\pi(G_1) \cup \{\pi(Q_1)\}$ determine a unique cubic curve since any point not in $\pi(G_1)$ imposes one condition on this pencil. Now consider G_2. Its projection also defines a pencil, and it contains 7 points of G_1 together with Q_1 and Q_2. Thus the cubic passing through $\pi(G_2) \cup \pi(P_{11})$ must be the same cubic passing through $\pi(G_1) \cup \pi(Q_1)$ (and then also Q_2). In other words,

$$\pi(G_1 \setminus \{P_{32}\}) \cup \{\pi(Q_1)\} = \pi(G_2 \setminus \{Q_2\}) \cup \{\pi(P_{11})\}$$

and by Cayley-Bacharach P_{33} and Q_2 also are in the same such cubic. (Repeating the same argument with G_3 we see that this cubic also contains Q_3.)

As an application of Bezout's Theorem, note that such a cubic curve has no linear components, and that it has no components in common with the quartic curve mentioned above. This is because if the cubic contains a line L, then L contains at most three points of $\pi(X_{D_4})$ so there would be at least 9 points on a conic. But there are too many sets of three collinear points.

In [POLITUS, Theorem 4.10], the authors show that X_{D_4} is, up to projectivities, the only non-trivial non-grid $(3, b)$-geproci set. However, for any values of $4 \leq a \leq b$ there is a non-degenerate and non-grid (a, b)-geproci set – see [POLITUS, Theorem 4.2]. Many questions about geproci sets are still open; see Chapter 8 of the mentioned paper for a list of open questions.

12.4. **Weddle locus.** As seen in Exercise 63, given a finite set $Z \subseteq \mathbb{P}^3$ and a degree d, it is often true that there is not even one degree d cone which contains Z when the vertex is a general point. In such cases there still can be a nonempty locus of points occurring as the vertex of a degree d cone containing Z. Studying such vertex loci is of interest in its own right, but will also be related to the Lefschetz properties.

Example 12.13. Let us begin by illustrating an issue that we will have to deal with when we make our definitions.

Let Z_1 be a set of 6 points in $\mathbb{P}^3$ in linear general position, and let Z_2 be a set of 6 points consisting of 3 points on one line, λ_1, and 3 points on a different line, λ_2, disjoint from the first one. Both Z_1 and Z_2 have h-vector $(1, 3, 2)$ and thus impose independent conditions on quadrics. Both also lie on a 4-dimensional (vector space dimension) family of quadrics.

We will see shortly that a general projection of Z_1 is a set of 6 points in $\mathbb{P}^2$ not lying on a conic, while clearly a general projection of Z_2 does lie on a conic (namely a union of two lines).

In this section we will be interested in keeping track of special projections. We will see that there is a quartic surface in $\mathbb{P}^3$, the Weddle surface, consisting of the locus of points from which the projection takes Z_1 to 6 points on a conic. But what are we to make of Z_2? There are two points of view.

First, we could say that since the general projection lies on a unique conic, the thing to look for is the locus of points from which the projection lies on a *pencil* of conics. This would be $\lambda_1 \cup \lambda_2$, since three points get collapsed to one. But a different point of view is that since we expect 6 points in the plane to lie on no conic, *all* projections are special. This latter point of view meshes better with the Lefschetz connection that we will come to soon so it is what we will use for our definition below, but note that for instance [POLITUS] took the former point of view. (For Z_1 there is no such distinction.)

Let $Z = \{P_1, \dots, P_r\} \subset \mathbb{P}^n$ be a set of distinct points. Let $H \cong \mathbb{P}^{n-1}$ be a general hyperplane. Let P be a point not in Z and let $\pi_P : \mathbb{P}^n \backslash \{P\} \to H$ be the projection from P. Let d be a positive integer. The homogeneous component $[I_Z \cap I_P^d]_d$ in degree d is the $\mathbb{C}$-vector space span of all forms of degree d that vanish on Z and vanish to order d (or more) at P. You should convince yourself that

$$\dim[I_Z \cap I_P^d]_d = \dim[I_{\pi_P(Z)}]_d$$

(where the first ideal is in $\mathbb{C}[x_0, \ldots, x_n]$ and the second is in $\mathbb{C}[X_0, \ldots, x_{n-1}]$) and that the elements of $[I_Z \cap I_P^d]_d$ are cones with vertex at P.

Let

$$\delta(Z, d) = \max\left\{ \binom{d+n-1}{n-1} - |Z|, 0 \right\}.$$

Note that $\delta(Z, d)$ is the minimum possible value of $\dim[I_{\pi_p(A)}]_d$. Achieving it means that the r points of $\pi_p(Z)$ impose independent conditions on forms of degree d in H for as long as numerically possible. In our setting, almost always this minimum will be the first of the two possibilities.

Definition 12.14. The *d-Weddle locus* of Z is the closure of the set of points $P \in \mathbb{P}^n \setminus Z$ (if any) for which $\dim_k[I_Z \cap I(P)^d]_d$ does not achieve its minimum:

$$\dim[I_Z \cap I_P^d]_d = \dim[I_{\pi_P(Z)}]_d > \delta(Z, d).$$

Thus

$$\mathcal{W}_d(Z) = \overline{\{P \in \mathbb{P}^n \mid \dim[I_{\pi_P(Z)}]_d > \delta(Z, d)\}}.$$

Example 12.15. Let us return to the situation of Example 12.13. If $Z_1 \subset \mathbb{P}^3$ is a set of 6 points in linear general position (see Exercise 63) then the general projection of Z_1 does not lie on a conic. Note that $\delta(Z_1, 2) = 6 - 6 = 0$. So, the 2-Weddle locus, which is known as *Weddle surface*, is the closure of the locus of points $P \notin Z_1$ in $\mathbb{P}^3$ that are the vertices of quadric cones in $\mathbb{P}^3$ containing Z_1. Equivalently, the Weddle surface is the closure of the locus of points $P \notin Z_1$ from which Z_1 projects to a set $\pi_P(Z_1) \subset \mathbb{P}^2$ contained in a conic. We will justify shortly the use of the word "surface" here, and see that $\mathcal{W}_2(Z_1)$ is a surface of degree 4.

What happens with Z_2? We still have $\delta(Z_2, 2) = 0$, but now for any $P \in \mathbb{P}^3$ we have

$$\dim[I_{Z_2} \cap I_P^2]_2 = \dim[I_{\pi_P(Z_2)}]_2 > 0$$

so the Weddle locus $\mathcal{W}_2(Z_2) = \mathbb{P}^3$.

As we indicated above, it is classically known that the Weddle surface has degree 4. There are several ways to construct the equations of the d-Weddle locus of a set of reduced points Z; in these notes we describe an approach based on Macaulay duality. This will give us the fact that for six points in linear general position the 2-Weddle locus is a surface of degree 4, and it will also finally give us our connection with the Lefschetz properties, and specifically with a certain non-Lefschetz locus (see section 9 for the definition).

12.5. **Macaulay duality.** Consider the polynomial rings

$$R = \mathbb{C}[x_0, \ldots, x_n] = \mathbb{C}[\mathbb{P}^n] \text{ and } R^* = \mathbb{C}[\partial_{x_0}, \ldots, \partial_{x_n}] = \mathbb{C}[(\mathbb{P}^n)^*],$$

where formally we think of the differential operators ∂_{x_i} as independent indeterminates.

Macaulay duality comes from regarding R^* as acting on R. Given a point $P = [p_0 : \ldots : p_n] \in \mathbb{P}^n$, the dual of P, denoted by P^* is the hyperplane in $(\mathbb{P}^n)^*$ defined by the linear form $L_P = \sum p_i \partial_{x_i} \in [R^*]_1$. The form L_P is the annihilator of $[I_P]_1$, i.e., as vector spaces $[I_P]_1$ is isomorphic to $[R^*/(L_P)]_1$.

(For example, when $n = 3$ let $P = [0, 0, 0, 1]$, $I_P = (x_0, x_1, x_2)$, $L_P = \partial_{x_3}$. We have that $[I_P]_1$ is annihilated by L_P and $\dim[\mathbb{C}[\partial_{x_0}, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}]/L_P]_1 = \dim[I_P]_1$.)

More generally for integers $0 \le k \le t$, the annihilator of $[I_P^k]_t$ under this action is $[({L_P}^{t-k+1})]_t$, hence we have the following isomorphism of vector spaces

$$[I_P^k]_t \cong [R^*/({L_P}^{t-k+1})]_t,$$

which can be applied to $[I_Z \cap I_P^d]_d$, $Z = \{P_1, \dots, P_r\}$ to get

$$[I_Z \cap I_P^d]_d = [I_{P_1} \cap \dots \cap I_{P_r} \cap I_P^d]_d \cong [R^*/(L_{P_1}^d, \dots, L_{P_r}^d, L_P)]_d.$$

Considering the following exact sequence

$$\left[\frac{R^*}{(L_{P_1}^d, \dots, L_{P_r}^d)}\right]_{d-1} \xrightarrow{\times L_P} \left[\frac{R^*}{(L_{P_1}^d, \dots, L_{P_r}^d)}\right]_d \to \left[\frac{R^*}{(L_{P_1}^d, \dots, L_{P_r}^d, L_P)}\right]_d \to 0 \qquad (12.1)$$

where $\times L_P$ denotes the map given by multiplication by L_P. We get

$$\operatorname{coker}(\times L_P) \cong [I_Z \cap I_P^d]_d.$$

But we saw above that

$$[I_Z \cap I_P^d]_d \cong [I_{\pi_P(Z)}]_d.$$

So looking for the set of points P for which the projection lies on unexpectedly many hypersurfaces (in H) of degree d is equivalent to looking for the set of points P for which $\times L_P$ has unexpectedly small rank! We conclude:

> *The d-Weddle locus for a set of points $Z \subset \mathbb{P}^n$ is equal to the non-Lefschetz locus for the algebra $R^*/(L_{P_1}^d, \dots, L_{P_r}^d)$ from degree $d-1$ to degree d. Note that this locus may be all of $\mathbb{P}^n$.*

We will return to this connection shortly.

Now we want to give a scheme structure to the d-Weddle locus. Denote by $A_d(Z)$ the matrix associated to $\times L_P$ (after a choice of basis). Then the d-Weddle locus of Z is the closure of the locus of points P such that $\operatorname{rank}(A_d(Z))$ is lower than expected. So, the ideal of the maximal minors of $A_d(Z)$ gives an ideal which defines the d-Weddle locus of Z, with eventually either some embedded components or non reduced components.

Definition 12.16. The d-Weddle scheme of Z is the scheme defined by saturation of the ideal of the maximal nonzero minors of $A_d(Z)$.

Example 12.17. We again return to the situation of Example 12.13. Consider Z_1. Notice that $n = 3$, $d = 2$ and

$$\dim \left[\frac{R^*}{(L_{P_1}^2, \dots, L_{P_6}^2)}\right]_1 = 4 \quad \text{and} \quad \dim \left[\frac{R^*}{(L_{P_1}^2, \dots, L_{P_6}^2)}\right]_2 = 10 - 6 = 4. \qquad (12.2)$$

(For the last calculation, linear general position forces the h-vector of Z_1 to be $(1, 3, 2)$ so the six points impose independent conditions on forms of degree 2 in $\mathbb{P}^3$.) Then the Macaulay duality matrix defining the 2-Weddle locus is a 4×4 matrix A_2 of linear forms. Since the general projection does not lie on a conic, we see that the determinant of A_2 is not zero so it defines a quartic surface as claimed.

For Z_2, the dimensions obtained in (12.2) are the same as for Z_1. However, now the cokernel is at least 1-dimensional for all P, so the determinant of A_2 must be zero. Hence the 2-Weddle locus is all of $\mathbb{P}^3$.

Let us examine this using coordinates. Now Z_2 consists again of six points but in a $(2,3)$-grid:

$$Z = \left\{ \begin{array}{l} [1:0:0:0], [0:1:0:0], [1:1:0:0], \\ [0:0:1:0], [0:0:0:1], [0:0:1:1] \end{array} \right\}.$$

The Macaulay duality matrix defining the 2-Weddle locus of Z is

$$\Gamma_2(Z) = \begin{pmatrix} z & 0 & x & 0 \\ w & 0 & 0 & x \\ 0 & z & y & 0 \\ 0 & w & 0 & y \end{pmatrix}$$

which has determinant equal to zero (this is consistent with the fact that Z is geproci and its general projection lies on a conic, so the 2-Weddle locus is all of $\mathbb{P}^3$).

Let us examine this further, recalling the other perspective mentioned in Example 12.13. The ideal of submaximal minors of $A_2(Z_2)$ is

$$I = (xzw, xw^2, yzw, yw^2, xz^2, xzw, yz^2, yzw, xyz, xyw, y^2z, y^2w, x^2z, x^2w, xyz, xyw)$$

whose primary decomposition is

$$(y,x) \cap (w,z) \cap (w^2, z^2, y^2, x^2, yzw, xzw, xyw, xyz).$$

The ideal I is not saturated. The saturation of such ideal defines the two lines containing Z. So the projection of Z is contained in a pencil of conics only if we project from the points of these two lines, as predicted.

Example 12.18. Let $Z_3 = Y \cup \{Q\} \subset \mathbb{P}^3$ be a set of six points such that Y consists of five general points in a plane H, and Q is a general point in $\mathbb{P}^3$.

Let $\mathcal{Q}$ be the quadric cone with vertex Q over the conic C in H defined by the five points. Notice that a general projection of Z_3 does not lie on a conic, since that conic would have to be the projection of C but there is no reason for Q to be mapped to this conic. So the 2-Weddle scheme is not all of $\mathbb{P}^3$.

On the other hand, one checks that the dimensions from (12.2) continue to hold here (in fact the h-vector of Z_3 is $(1,3,2)$ again). So A_2 continues to be a 4×4 matrix of linear forms with nonzero determinant.

From the geometry of the situation we note that the 2-Weddle scheme is a proper subscheme of $\mathbb{P}^3$ supported on H and $\mathcal{Q}$. Indeed, it has only two components, the quadric $\mathcal{Q}$ and the plane H. In fact, the projection from any point not on either $\mathcal{Q}$ or H sends Z to six points not on a conic.

Putting all this together, the 2-Weddle scheme is a quartic surface determined by the determinant of a 4×4 matrix of linear forms, but this quartic is not reduced and it must have a double structure on H.

12.6. **Connection to WLP.** We conclude these notes by returning to the connection between Weddle loci and the non-Lefschetz locus in slightly more detail, and also about the Weak Lefschetz Property itself.

Let $A = R^*/I$ be an artinian graded algebra with the WLP. Recall from Section 9 that the non-Lefschetz locus of A is

$$\mathfrak{L}_A = \{P \in \mathbb{P}^n \mid L_P \text{ is not a Lefschetz element}\} \subseteq \mathbb{P}^n.$$

The set $\mathfrak{L}_A$ has a natural stratification given by the sets

$$\mathfrak{L}_{A,d} = \{P \in \mathbb{P}^n \mid \times L_P : [A]_{d-1} \to [A]_d \text{ does not have maximal rank}\}.$$

By Macaulay duality, given a set of reduced points $Z = \{P_1, \ldots, P_r\} \subseteq \mathbb{P}^n$ such that $A = R^*/(L^d_{P_1}, \ldots, L^d_{P_r})$ is a Weak Lefschetz Algebra, we have noted that the dimension of the last vector space in the exact sequence (12.1) is equal to $\dim[I_{P_1} \cap \cdots \cap I_{P_r} \cap I^d_P]_d$, so this is larger than expected if and only if the rank of $A_d(Z)$ is smaller than expected. Then $\mathfrak{L}_{A,d}$ is precisely the d-Weddle locus of Z.

For the quotient algebra $R^*/(L^d_{P_1}, \ldots, L^d_{P_r})$, notice that $\left[\frac{R^*}{(L^d_{P_1},\ldots,L^d_{P_r})}\right]_{d-1} = [R^*]_{d-1}$. From the exact sequence (12.1), we also see that in correspondence to a set of reduced points $Z = \{P_1, \ldots, P_r\}$, the failure of the Weak Lefschetz Property from degree $d-1$ to degree d is equivalent to having a-dim$(Z, d, d) >$ e-dim(Z, d, d), i.e. to the existence of an unexpected cone of degree d for Z.

Therefore, from [CM, Theorem 3.5] (which ensures that (a, b)-grids have unexpected cones in degree a, and also in degree b provided $a, b \geq 3$) we have the following result.

Proposition 12.19. *Let Z be an (a, b)-grid with $b \geq a \geq 2$ and $b \geq 3$, and let $L_1, \ldots, L_{ab}$ be the dual linear forms. Then $R/(L^a_1, \ldots, L^a_{ab})$ fails the Weak Lefschetz Property from degree $a-1$ to degree a, and if $b \geq a \geq 3$ then $R/(L^b_1, \ldots, L^b_{ab})$ fails the Weak Lefschetz Property from degree $b-1$ to degree b.*

Furthermore, from [POLITUS, Chapter 7], (a, b)-geproci sets of points admit unexpected cones of degree a and almost always also of degree b. Thus any such result about geproci sets of points gives an example of failure of the Weak Lefschetz Property.

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13. Solutions to exercises

Exercise 1.

We want to count monomials. Imagine a set of $d+n-1$ objects, places side by side. Among these, choose $n-1$ of them, so there remain d unchosen objects. These remaining d unchosen objects will represent variables. To the left of the first marker, the number of objects represents the power of x_1 in the monomial. Between the first and the second, the number of objects represents the power of x_2. And so on. Each monomial corresponds to a unique choice of markers, and each choice of markers corresponds to a unique monomial.

For example, suppose $n = 4$ and $d = 5$. We want to choose 3 markers from a set of 8 objects. Below, we have 8 objects, of which the bullets $\bullet$ represent the choice of 3 and the $\times$ represent unchosen objects.

$$\times \quad \bullet \quad \bullet \quad \times \quad \times \quad \times \quad \times \quad \bullet$$

To the left of the first marker is one object $\times$, so the monomial contains x_1^1. Between the first and the second are no $\times$, so there is no power of x_2. then we have four $\times$, so we have x_3^4. Finally, there is no power of x_4. So this choice corresponds to $x_1x_3^4$. □

Exercise 2.

(a) It's clear that $x+y \in \langle x, y\rangle$ and $x-y \in \langle x, y\rangle$ so we have $\langle x+y, x-y\rangle \subset \langle x, y\rangle$. For the reverse inclusion, we have $x = \frac{1}{2}[(x+y)+(x-y)]$ and $y = \frac{1}{2}[(x+y)-(x-y)]$.

(b) For the first equality, one inclusion is clear, namely $\supseteq$. So we want to show that
$$x, y \in \langle x+xy, y+xy, x^2, y^2\rangle.$$
In fact,
$$x = (1-y)(x+xy) + (x)(y^2)$$
and
$$y = (1-x)(y+xy) + (y)(x^2).$$
For the other equality, we'll instead show that
$$\langle x, y\rangle = \langle x+xy, y+xy, x^2\rangle.$$
Again $\supseteq$ is clear, so we'll show that both x and y are in the ideal on the right. First,
$$x = (x+xy) - x(y+xy) + y(x^2).$$
So we can (and will) freely use the fact that x is in this ideal. Then
$$y = (y+xy) - y(x).$$

(c) First we show that
$$\langle x, y\rangle \neq \langle x+xy, y+xy\rangle.$$
It's enough to show that $x \notin \langle x+xy, y+xy\rangle$. Notice that
$$x-y = (x+xy) - (y+xy)$$
so it's enough to show that
$$\langle x-y, x+xy\rangle \neq \langle x, y\rangle.$$
Note that $x+xy = x(1+y)$. Suppose
$$A(x-y) + Bx(1+y) = x.$$

Set $y = x$. Then we have

$$B(x,x)x(1+x) = x.$$

This is impossible by degree considerations.

Now let's show that

$$\langle x + xy, x^2 \rangle \neq \langle x, y \rangle.$$

Suppose

$$Ax(1+y) + Bx^2 = x.$$

Then

$$A(1+y) + Bx = 1.$$

Now set $x = 0$. We get $A(0,y)(1+y) = 1$, which again is impossible for degree reasons.

Finally, let's show that

$$\langle y + xy, x^2 \rangle \neq \langle x, y \rangle.$$

Suppose we have $A(y + xy) + Bx^2 = x$. Set $y = 0$. We get

$$B(x,0)x^2 = x,$$

which is impossible. □

Exercise 3.

We prove both inclusions. Let $P \in V \cap W$. Since $P \in V$, $f_i(P) = 0$ for all $1 \leq i \leq s$. Since $P \in W$, $g_j(P) = 0$ for all $1 \leq j \leq t$. Thus $P \in \mathbb{V}(f_1, \dots, f_s, g_1, \dots, g_t)$.

Now assume $P \in \mathbb{V}(f_1, \dots, f_s, g_1, \dots, g_t)$. In particular, $P \in \mathbb{V}(f_1, \dots, f_s) = V$ and $P \in \mathbb{V}(g_1, \dots, g_t) = W$, so $P \in V \cap W$. □

Exercise 4.

We first prove that a single point in $\mathbb{A}^n$ is an affine variety. Indeed, if $P = (a_1, a_2, \dots, a_n) \in \mathbb{A}^n$ then

$$P = \mathbb{V}(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n).$$

Now let $V = \{P_1, P_2, \dots, P_m\}$. By what we have just seen, each P_i is, by itself, an affine variety. So we proceed by induction on the number of points, having just proven the case of one point. Assume that the statement is true for $m - 1$ points, i.e. any subset of all but one point of V. So for example, let

$$X = \{P_1, \dots, P_{m-1}\}$$

and note that $V = X \cup P_m$. By induction, X is an affine variety. As noted, P_m is also an affine variety. So by Lemma 2 of [CLO], $V = X \cup P_m$ is also an affine variety. □

Exercise 5.

(a) In particular we have $f(n, 0) = 0$ for all $n \in \mathbb{Z}$. But $g(x) = f(x, 0)$ is a polynomial, and the first sentence means that $g(x)$ has infinitely many zeros. So $g(x)$ is the zero polynomial.

This means that plugging in $y = 0$ into $f(x, y)$ gives the zero polynomial, so $f(x, y)$ contains no terms that are pure powers of x. In a similar way we can show that $f(x, y)$ contains no terms that are pure powers of y.

Now consider $f(x,1)$. Since each term of $f(x,y)$ contains both powers of x and of y, $f(x,1)$ converts each term of $f(x,y)$ into a term involving only x. Now the fact that $f(x,1)$ has infinitely many zeros means that it, too, is the zero polynomial, so all its terms are zero. This means that all terms of $f(x,y)$ are zero, so f is the zero polynomial.

(b) From (a), if $f \in I(Z)$ then f is the zero polynomial. If Z were an affine variety then we would have $Z = \mathbb{V}(f_1, \dots, f_s)$ for some polynomials $f_1, \dots, f_s$ that (by definition) vanish on Z. But any polynomial vanishing on Z is the zero polynomial, so the smallest variety containing Z is $\mathbb{R}^2$. In particular, Z is not an affine variety. □

Exercise 6.

Consider the following statement:

If $f(x,y)$ is a polynomial that vanishes at each point of X then f vanishes on the whole curve $x^3 - y + 1 = 0$. (∗)

We claim that proving (∗) will guarantee that X is not an affine variety.

Indeed, let C be the curve $\mathbb{V}(x^3 - y + 1) \subset \mathbb{R}^2$. Notice that C contains points that are not on X, for example the point $(\pi, \pi^3 + 1)$. Suppose it were true that X were an affine variety, so $X = \mathbb{V}(f_1, \dots, f_s)$ for some polynomials $f_1, \dots, f_s \in \mathbb{R}[x,y]$. That means that

the common vanishing locus of $f_1, \dots, f_s$ is precisely X. (∗∗)

If every polynomial f that vanishes at all points of X also vanishes on all of C, then this is true of $f_1, \dots, f_s$, so (∗∗) can't be true – the common vanishing locus contains a lot of other points, such as $(\pi, \pi^3 + 1)$. So this contradiction shows that X is not an affine variety.

So we just have to prove (∗). Again by contradiction. Suppose $f \in \mathbb{R}[x,y]$ vanishes at every point of X (i.e. $X \subset \mathbb{V}(f)$).

Consider the intersection of $\mathbb{V}(f)$ and $\mathbb{V}(x^3 - y + 1)$. By Lemma 2, this intersection is an affine variety:

$$\mathbb{V}(f) \cap \mathbb{V}(x^3 - y + 1) = \mathbb{V}(f, x^3 - y + 1).$$

Notice that $X \subset \mathbb{V}(f) \cap \mathbb{V}(x^3 - y + 1)$. This intersection is the set of points $(a,b) \in \mathbb{R}^2$ such that

$$f(a,b) = 0 \quad \text{and} \quad a^3 - b + 1 = 0.$$

The second of these equations says that for a point in this intersection, $b = a^3 + 1$. The first of the equations then says that any of these intersection points satisfies

$$f(a, a^3 + 1) = 0.$$

The fact that $X \subset \mathbb{V}(f) \cap \mathbb{V}(x^3 - y + 1)$ means that the above equation is satisfied whenever $a \in \mathbb{Z}$.

But $f(t, t^3 + 1)$ is a polynomial in one variable, t. The fact that it vanishes whenever t is an integer says that it has infinitely many roots or else is the zero polynomial. But a non-zero polynomial in one variable has finitely many roots. Thus $f(t, t^3 + 1)$ is the zero polynomial. This means that f vanishes at any point (x,y) such that $y = x^3 + 1$, i.e. it vanishes on the whole curve $\mathbb{V}(x^3 - y + 1)$. □

Exercise 7.

Assume that V is a subvariety of k^1. Then V is the vanishing locus of a set of polynomials in $k[x]$. Now, one single polynomial $f \in k[x]$ has at most finitely many roots, so even $\mathbb{V}(f)$ is a finite set of points. Adding additional polynomials can only make the common vanishing locus smaller, so we are done.

Conversely, assume that V is a finite set of ℓ points. Since $V \subset k^1$, each point of V can be viewed as an element $a_i \in k$, $1 \leq i \leq \ell$, so $V = \{a_1, \dots, a_\ell\}$. Thus

$$V = \mathbb{V}((x - a_1)(x - a_2) \cdots (x - a_\ell))$$

is a subvariety of k^1. □

Exercise 8.

(a) Since $\mathbb{F}_2 = \{0, 1\}$, notice that if either a or b is 0 we are done. The only other case is $a = b = 1$, and this reduces to $1 - 1 = 0$.

(b) One solution is $x_1^2 \dots x_n^2 - x_1 \dots x_n$. As before, if any $a_i = 0$ we are done, and the only other possibility is $a_i = 1$ for all i, in which case we have $1 - 1 = 0$.

(c) Fermat's theorem says that $a^p = a$ for all $a \in \mathbb{F}_p$, so as before one solution is $x_1^p \dots x_n^p - x_1 \dots x_n$. □

Exercise 9.

(a) Let $P \in S$. Let $f \in \mathbb{I}(S)$. By definition, $f(P) = 0$. This is true for every $f \in \mathbb{I}(S)$. Hence by definition, $P \in \mathbb{V}(\mathbb{I}(S))$.

(b) Let S_1 be the indicated set. We want to compute $\mathbb{I}(S_1)$. Let $f \in \mathbb{I}(S_1)$. So $f(0, m) = 0$ for all $m \in \mathbb{Z}$. Note that f has some degree, say d. Write f in the form

$$f(x, y) = a_0 + [a_{1,0}x + a_{0,1}y] + [a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2] + \cdots + [a_{d,0}x^d + a_{d-1,1}x^{d-1}y + \cdots + a_{0,d}y^d]$$

(the subscripts just tell you what monomial they correspond to). We are interested in plugging in the points $(0, m)$ for all m. Let's do it in two steps, first plugging in $x = 0$. We get that

$$f(0, y) = a_0 + a_{0,1}y + a_{0,2}y^2 + \cdots + a_{0,d}y^d$$

is a polynomial in one variable that has infinitely many roots. Since $\mathbb{R}$ is an infinite field, this must be the zero polynomial, i.e.

$$a_0 = a_{0,1} = \cdots = a_{0,d} = 0.$$

But with these coefficients being 0, it means that f is divisible by x. Thus $\mathbb{I}(S_1) \subset \langle x \rangle$. On the other hand, clearly any element of $\langle x \rangle$ vanishes at every point of S_1, so we have the reverse inclusion, and

$$\mathbb{I}(S_1) = \langle x \rangle.$$

But then

$$\mathbb{V}(\mathbb{I}(S_1)) = \mathbb{V}(\langle x \rangle) = \{(a, b) \in \mathbb{R}^2 \mid a = 0\},$$

that is, $\mathbb{V}(\mathbb{I}(S_1))$ is the y-axis, which properly contains S_1.

(c) We showed in (a) that $S \subset \mathbb{V}(\mathbb{I}(S))$, so we just have to prove the reverse inclusion. Since S is a variety, we are assuming that there are polynomials $f_1, \dots, f_s$ such that $S = \mathbb{V}(f_1, \dots, f_s)$. But then we also have $S = \mathbb{V}(\langle f_1, \dots, f_s \rangle)$. By definition, each f_i vanishes at every point of S, so $\langle f_1, \dots, f_s \rangle \subseteq \mathbb{I}(S)$. By the inclusion-reversing property, we conclude

$$S = \mathbb{V}(f_1, \dots, f_s) = \mathbb{V}(\langle f_1, \dots, f_s \rangle) \supseteq \mathbb{V}(\mathbb{I}(S)),$$

which is what we wanted to prove. □

Exercise 10.

Let $f \in k[x_1, \dots, x_n]$ be a polynomial such that $f^m \in \mathbb{I}(V)$. This means that $f^m(P) = f(P)^m = 0$ for all $P \in V$. But $f(P)$ is an element of the field k, and if a power of a field element is zero then that element is itself zero (because a field is, in particular, an integral domain). Thus $f(P) = 0$ for all $P \in V$, so $f \in \mathbb{I}(V)$. □

Exercise 11.

(a) We use the fact that both I and J are ideals. Since $0 \in I$ and $0 \in J$, we have $0 \in I \cap J$. If $f, g \in I \cap J$ then f and g are both in I and both in J, so $f + g \in I \cap J$. If $f \in I \cap J$ and $h \in R$ then $hf \in I$ and $hf \in J$ so $hf \in I \cap J$.

(b) $0 \in I$ and $0 \in J$ so $0 = 0 \cdot 0 \in IJ$. Assume $f = \sum_{i=1}^{m} f_i g_i$ for some $f_i \in I, g_i \in J$ and $g = \sum_{i=1}^{m'} f_i' g_i'$ for some $f_i' \in I, g_i' \in J$. Then

$$f + g = \sum_{i=1}^{m} f_i g_j + \sum_{i=1}^{m'} f_i' g_j' \in IJ.$$

Finally, if $f = \sum_{i=1}^{m} f_i g_i$ for some $f_i \in I, g_i \in J$ and $h \in R$ then

$$hf = h \cdot \sum_{i=1}^{m} f_i g_i = \sum_{i=1}^{m} (hf_i) g_i \in IJ$$

since $hf_i \in I$ (because I is an ideal).

(c) It's enough to prove that each generator of IJ is in $I \cap J$ (why?). If $I = \langle f_1, \dots, f_s \rangle$ and $J = \langle g_1, \dots, g_t \rangle$ then the generators of IJ have the form $f_i g_j$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. But then $f_i g_j \in I$ (since $f_i \in I$) and also $f_i g_j \in J$ (since $g_j \in J$) so $f_i g_i \in I \cap J$.

(d) For example take $I = \langle x \rangle$ and $J = \langle x \rangle$. Then $I \cap J$ is clearly equal to $\langle x \rangle$, while $IJ = \langle x^2 \rangle$. We have already seen that these two ideals are not equal.

(e) Let's prove the two inclusions.

$\subseteq$:

Let $P \in \mathbb{V}(IJ)$. We want to show that $P \in \mathbb{V}(I) \cup \mathbb{V}(J)$. If $P \in \mathbb{V}(I)$ then we're done, so assume $P \notin \mathbb{V}(I)$; we want to show that then $P \in \mathbb{V}(J)$. Since $P \notin \mathbb{V}(I)$, there is some $f \in I$ such that $f(P) \neq 0$. But $fg \in IJ$ for all $g \in J$; hence $(fg)(P) = 0$ for all $g \in J$. Thus $P \in \mathbb{V}(J)$ as desired.

$\supseteq$:

Let $P \in \mathbb{V}(I) \cup \mathbb{V}(J)$. So either $P \in \mathbb{V}(I)$ or $P \in \mathbb{V}(J)$ (or both). Assume without loss of generality that $P \in \mathbb{V}(I)$. Then $f(P) = 0$ for all $f \in I$. Let $g \in IJ$, so

$$g = \sum_{i=1}^{m} f_i g_i \mid f_i \in I \text{ and } g_i \in J.$$

Then we get $g(P) = \sum_{i=1}^{m} f_i(P) g_i(P) = 0$. Hence $g(P) = 0$ for all $g \in IJ$, and so $P \in \mathbb{V}(IJ)$.

(f) Again we prove the two inclusions.

$\subseteq$:

Let $P \in \mathbb{V}(I \cap J)$, so $h(P) = 0$ for all $h \in I \cap J$. Suppose that $P \notin \mathbb{V}(I)$. We want to show $P \in \mathbb{V}(J)$, i.e. we want to show that $g(P) = 0$ for all $g \in J$. Since $P \notin \mathbb{V}(I)$, there is some $f \in I$ such that $f(P) \neq 0$. Then for any $g \in J$, we know that $fg \in I \cap J$ so $(fg)(P) = 0$. Since$f(P) \neq 0$, this forces $g(P) = 0$ for all $g \in J$, so $P \in \mathbb{V}(J)$ as desired.

$\supseteq$:

Let $P \in \mathbb{V}(I) \cup \mathbb{V}(J)$, so either $P \in \mathbb{V}(I)$ or $P \in \mathbb{V}(J)$ or both. Let $f \in I \cap J$. Since f is in both I and J, we must have $f(P) = 0$. So $P \in \mathbb{V}(I \cap J)$. □

Exercise 12.

$\subseteq$:

Let $P = (a_1, \dots, a_n) \in \phi^{-1}(X)$, so $\phi(P) \in X$. This means

$$(F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)) \in X.$$

But $X = \mathbb{V}(G_1, \dots, G_k)$, so for any i with $1 \leq i \leq k$ we have

$$G_i(F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)) = 0.$$

That is, the polynomial $G_i(F_1, \dots, F_m)$ vanishes at P for $1 \leq i \leq k$, so

$$P \in \mathbb{V}(G_1(F_1, \dots, F_m), \dots, G_k(F_1, \dots, F_m))$$

as desired.

$\supseteq$:

Let

$$P = (a_1, \dots, a_n) \in \mathbb{V}(G_1(F_1, \dots, F_m), \dots, G_k(F_1, \dots, F_m)).$$

This means $(F_1(P), \dots, F_m(P)) \in \mathbb{V}(G_1, \dots, G_k) = X \subset \mathbb{C}^m$. But $(F_1(P), \dots, F_m(P)) = \phi(P)$, so $\phi(P) \in X$, i.e. $P \in \phi^{-1}(X)$ as desired. □

Exercise 13.

First we find a function $\phi : k[x_1, \dots, x_{n-1}][x_n] \to k[x_1, \dots, x_n]$. If $f \in k[x_1, \dots, x_{n-1}][x_n]$, notice that

$$f = g_0(x_1, \dots, x_{n-1}) + g_1(x_1, \dots, x_{n-1})x_n + \cdots + g_d(x_1, \dots, x_{n-1})x_n^d$$

for some non-negative integer d. So f can be viewed naturally as an element of $k[x_1, \dots, x_n]$ just by multiplying out all the terms. Define $\phi(f) = f$ in this way.

Now note that ϕ is a ring homomorphism. Indeed, $\phi(f + g) = \phi(f) + \phi(g) = f + g$ and $\phi(fg) = \phi(f)\phi(g) = fg$ are both immediate from the definition.

Next notice that ϕ is injective: again from the definition, $f \in \ker\phi$ if and only if $\phi(f) = 0$ if and only if $f = 0$.

Finally notice that ϕ is surjective: by separating out the x_n's, any polynomial in $k[x_1, \dots, x_n]$ can be expressed as a polynomial in $k[x_1, \dots, x_{n-1}][x_n]$. □

Exercise 14.

Consider the chain of ideals

$$\langle f_1\rangle \subseteq \langle f_1, f_2\rangle \subseteq \langle f_1, f_2, f_3\rangle \subseteq \cdots$$

Since $k[w, x, y, z]$ is Noetherian, this chain stabilizes. That is, there is some N so that

$$\langle f_1, \dots, f_N\rangle = \langle f_1, \dots, f_N, f_{N+1}, \dots, f_j\rangle$$

for any $j \geq N + 1$. So in particular, each f_j can be written as a linear combination of $f_1, \dots, f_N$. □

Exercise 15.

(a) We claim that $f = f_1^2 + \cdots + f_s^2$ does the trick. First show $V \subseteq \mathbb{V}(f)$. If $P \in V$ then $f_i(P) = 0$ for all $1 \leq i \leq s$, so $f_i^2(P) = 0$ for all $1 \leq i \leq s$ and hence the sum $f(P) = 0$ as well.

Conversely, we'll show that $V \supseteq \mathbb{V}(f)$. Let $P \in \mathbb{V}(f)$, so

$$f(P) = (f_1^2 + \cdots + f_s^2)(P) = f_1^2(P) + \cdots + f_s^2(P) = 0.$$

But we are working over the real numbers, so each term of $f_1^2(P) + \cdots + f_s^2(P)$ is non-negative. Thus it can only equal zero if $f_1(P) = \cdots = f_s(P) = 0$, i.e. if $P \in V$.

(b) Let $f = f_1^2 + \cdots + f_s^2$, which is certainly in $I = \langle f_1, \dots, f_s\rangle$. From part a) we know that

$$\emptyset = \mathbb{V}(I) = \mathbb{V}(\langle f_1, \dots, f_s\rangle) = \mathbb{V}(f),$$

so f has no zeros in $\mathbb{R}^n$. □

Exercise 16.

Let $J = \mathbb{I}(V) + \mathbb{I}(W)$. We first claim that $\mathbb{V}(J) = \emptyset$. If $P \in \mathbb{V}(J)$ then in particular every element of $\mathbb{I}(V)$ vanishes at P and every element of $\mathbb{I}(W)$ vanishes at P. Thus $P \in V$ and $P \in W$, i.e. $P \in V \cap W$. This is impossible since $V \cap W = \emptyset$.

But now $\mathbb{C}$ is algebraically closed, so the Weak Nullstellensatz holds. This means

$$J = \mathbb{I}(V) + \mathbb{I}(W) = \langle 1\rangle,$$

so the desired result holds. □

Exercise 17.

Since $k[x_1, \dots, x_n]$ is Noetherian, $\sqrt{I}$ is finitely generated. Say

$$\sqrt{I} = \langle f_1, \dots, f_s\rangle.$$

In particular, each f_i is in $\sqrt{I}$. Define $m_1, \dots, m_s$ so that $f_i^{m_i} \in I$ for each i. Let $p = m_1 + \cdots + m_s$.

Let $f \in \sqrt{I}$, so we can write $f = g_1 f_1 + \cdots + g_s f_s$, where $g_i \in k[x_1, \dots, x_n]$. Then

$$f^p = (g_1 f_1 + \cdots + g_s f_s)^p.$$

Each term in the expansion of f^p is of the form

$$Bf_1^{i_1}f_2^{i_2}\cdots f_s^{i_s}$$

where B is some (ugly) polynomial and $i_1+i_2+\cdots+i_s = p = m_1+\cdots+m_s$. As in class, we claim that for at least one subscript k we have $i_k \geq m_k$. This is a sort of pigeon-hole principle – if i_k is always less than m_k, it is impossible for $i_1+i_2+\cdots+i_s = p = m_1+\cdots+m_s$. But if $i_k \geq m_k$ then $f_k^{i_k} \in I$. So every such term in the expansion of f^p is in I, hence $f^p \in I$. □

Exercise 18.

(a) We have seen that
$$\mathbb{V}(I)\cap\mathbb{V}(J) = \mathbb{V}(I+J).$$
Hence under our conditions, $\mathbb{V}(I)\cap\mathbb{V}(J) = \emptyset$, i.e. $\mathbb{V}(I)$ and $\mathbb{V}(J)$ are disjoint.

(b) It is always true that $IJ \subseteq I\cap J$ so we only have to prove the reverse inclusion. From our assumption we have that for some $f \in I$ and $g \in J$, $1 = f+g$. Let $h \in I\cap J$. We want to show that $h \in IJ$. Multiplying both sides of the equation $1 = f+g$ by h gives $h = fh+gh$. The fact that $f \in I$ and $h \in J$ means that $fh \in IJ$. The fact that $h \in I$ and $g \in J$ means that $gh \in IJ$. Thus $h \in IJ$.

(c) It is enough to take $I = J = \langle x\rangle$. □

Exercise 19.

(a) No! Suppose $f^m \in \mathbb{I}(X)$, so f^m vanishes at every point of X. Then clearly f vanishes at every point of X. Hence $f \in \mathbb{I}(X)$, so $J = \mathbb{I}(X)$ is radical.

(b) Yes! $\mathbb{I}(X)$ being prime means that X is irreducible, so let's take the simplest non-irreducible example: two points. Let $X = \{(0,0),(1,0)\} \subset \mathbb{R}^2$, and take $J = \mathbb{I}(X)$. I'm happy with this as your final answer. But if you also tell me that $J = \langle y, x(x-1)\rangle$, that's good too. Notice that $x\cdot(x-1) \in J$ but neither x nor $x-1$ is in J, so J is not prime.

(c) Yes! Let $R = k[x,y]$ and let $J = \langle x\rangle$. J is prime, but it is not maximal since $J \subset \langle x,y\rangle$, which is also prime.

(d) Yes! Take $J = \langle x^2\rangle \subset \mathbb{R}[x,y]$. Then $\mathbb{V}(J)$ is the y-axis in $\mathbb{R}^2$, which is irreducible. Then $\mathbb{I}(\mathbb{V}(J)) = \langle x\rangle$, which is prime. But J itself is not prime, since $x\cdot x \in J$ but $x \notin J$.

(e) Yes! Take $J = \langle x^2\rangle \subset \mathbb{R}[x,y]$. Then $\mathbb{V}(J)$ is the y-axis in $\mathbb{R}^2$. The polynomial $f = x$ has the desired property.

(f) No! This is the main point of the Strong Nullstellensatz. If $f \in \mathbb{I}(\mathbb{V}(J))$ then $f^m \in J$ for some $m \geq 1$. □

Exercise 20.

By the Hilbert Basis Theorem, I has a finite generating set: $I = \langle f_1,\dots,f_r\rangle$. Since $I \subset \sqrt{J}$, each $f_i \in \sqrt{J}$. Thus for each i there is a positive integer m_i such that $f_i^{m_i} \in J$. Now we look at different powers of I,

$$I^m = \underbrace{\langle f_1,\dots,f_r\rangle\cdot\langle f_1,\dots,f_r\rangle\cdot\cdots\cdot\langle f_1,\dots,f_r\rangle}_{m \text{ times}}.$$

This is generated by the polynomials obtained by taking m of the f_i (possibly repeating) and multiplying them. We want to show that if we choose m big enough, then every such generator is in J.

If you were to take $m = (m_1 - 1) + (m_2 - 1) + \cdots + (m_r - 1) = (\sum m_i) - r$, then it wouldn't quite work because you'd get

$$f_1^{m_1-1} \cdot f_2^{m_2-1} \cdot \cdots \cdot f_r^{m_r-1}$$

as one of the generators, which is not necessarily in J. However, let m be anything bigger than this, e.g. $m = (\sum m_i) - r + 1$. Then every generator of I^m is of the form

$$f_1^{a_1} \cdot f_2^{a_2} \cdot \cdots \cdot f_r^{a_r}$$

with $\sum a_i = m$, and this forces at least one of the a_i to be bigger than or equal to the corresponding m_i; thus every generator of I^m is in J. Hence $I^m \subset J$. □

Exercise 21.

(a) Since I and J are homogeneous ideals, we can find generators for each that are homogeneous. Say $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_t \rangle$. Then

$$I + J = \langle f_1, \ldots, f_s, g_1, \ldots, g_t \rangle$$

is generated by homogeneous polynomials, hence is a homogeneous ideal. □

(b) We'll use the other condition for an ideal to be homogeneous. Let $f \in I \cap J$. Write f as a sum of homogeneous polynomials, $f = f_d + f_{d-1} + \cdots + f_1 + f_0$. Since $f \in I$ and I is homogeneous, each $f_i \in I$. Similarly for J. Thus each $f_i \in I \cap J$, so $I \cap J$ is homogeneous. □

Exercise 22.

(a) Let $f(x, y, z) = x^3yz + 4x^2yz^2 + 5xyz^3$. Notice that $d = 5$. Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2yz + 8xyz^2 + 5yz^3 \\ \frac{\partial f}{\partial y} &= x^3z + 4x^2z^2 + 5xz^3 \\ \frac{\partial f}{\partial z} &= x^3y + 8x^2yz + 15xyz^2 \end{aligned}$$

Then

$$\begin{aligned} & x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} \\ = \ & x(3x^2yz + 8xyz^2 + 5yz^3) + y(x^3z + 4x^2z^2 + 5xz^3) + z(x^3y + 8x^2yz + 15xyz^2) \\ = \ & (3x^3yz + 8x^2yz^2 + 5xyz^3) + (x^3yz + 4x^2yz^2 + 5xyz^3) + (x^3yz + 8x^2yz^2 + 15xyz^3) \\ = \ & 5(x^3yz + 4x^2yz^2 + 5xyz^3). \end{aligned}$$

(b) We know that

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n). \tag{13.1}$$

Then differentiate on both sides with respect to λ.

$$\frac{\partial}{\partial \lambda} f(\lambda x_0, \dots, \lambda x_n) = d\lambda^{d-1} f(x_0, \dots, x_n). \tag{13.2}$$

Let's look at the left-hand side. For $0 \leq i \leq n$ let $u_i = \lambda x_i$.

$$\begin{aligned} \frac{\partial}{\partial \lambda} f(\lambda x_0, \dots, \lambda x_n) &= \sum_{i=0}^{n} \left(\frac{\partial f}{\partial u_i} \right) \left(\frac{\partial u_i}{\partial \lambda} \right) \\ &= \sum_{i=0}^{n} \left(\left. \frac{\partial f}{\partial x_i} \right|_{x_i = u_i} \right) \cdot x_i \\ &= \sum_{i=0}^{n} \lambda^{d-1} x_i \frac{\partial f}{\partial x_i} \end{aligned} \tag{13.3}$$

where we have used the fact that $\frac{\partial f}{\partial x_i}$ is homogeneous of degree $d-1$ and applied (13.1) to the partials. Now substitute the result of (13.3) into (13.2) and divide by λ^{d-1} (which is non-zero) to obtain the result.

(c) First note that

$$f_x = \frac{\partial f}{\partial x} = yz, \quad f_y = \frac{\partial f}{\partial y} = xz, \quad f_z = \frac{\partial f}{\partial z} = xy.$$

Now, $\mathbb{V}(f) = \mathbb{V}(xyz)$ is the union of the three lines defined by $x = 0$, $y = 0$ and $z = 0$. On the other hand, $\mathbb{V}(f_x, f_y, f_z)$ is the locus defined by

$$\begin{aligned} yz &= 0 \\ xz &= 0 \\ xy &= 0. \end{aligned}$$

A quick calculation reveals

$$\mathbb{V}(f_x, f_y, f_z) = \mathbb{V}(x, y) \cup \mathbb{V}(x, z) \cup \mathbb{V}(y, z).$$

This is precisely the union of the three points of pairwise intersection of the three lines in $\mathbb{V}(xyz)$, that is, the points $\{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$. In particular, $\mathbb{V}(f_x, f_y, f_z)$ is a subset of $\mathbb{V}(xyz)$. And indeed, since Euler's theorem gives, in this case, that

$$x \cdot f_x + y \cdot f_y + z \cdot f_z = 3 \cdot f,$$

if $P \in \mathbb{V}(f_x, f_y, f_z)$ then f_x, f_y, f_z all vanish at P, so Euler's theorem implies that f vanishes at P, so in particular $P \in \mathbb{V}(f)$.

(d) First note that $f = x^2yz + xy^2z + xyz^2$, so

$$f_x = \frac{\partial f}{\partial x} = 2xyz + y^2z + yz^2 = yz(2x + y + z)$$

$$f_y = \frac{\partial f}{\partial y} = x^2z + 2xyz + xz^2 = xz(x + 2y + z)$$

$$f_z = \frac{\partial f}{\partial z} = x^2y + xy^2 + 2xyz = xy(x + y + 2z).$$

Now, $\mathbb{V}(f) = \mathbb{V}(xyz(x+y+z))$ is the union of the four lines defined by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 0$. On the other hand, $\mathbb{V}(f_x, f_y, f_z)$ is the locus defined by

$$\begin{aligned} yz(2x + y + z) &= 0 \\ xz(x + 2y + z) &= 0 \\ xy(x + y + 2z) &= 0. \end{aligned}$$

Since each of these is a product of three linear forms, each equation is satisfied exactly when one (or more) of the factors is zero. Then for a point to be in the solution set $\mathbb{V}(f_x, f_y, f_z)$ we need one of the following lines to hold:

$$x = 0 \Rightarrow yz(y + z) = 0 \Rightarrow y = 0 \text{ OR } z = 0 \text{ OR } y = -z$$

$$y = 0 \Rightarrow xz(x + z) = 0 \Rightarrow x = 0 \text{ OR } z = 0 \text{ OR } x = -z$$

$$z = 0 \Rightarrow xy(x + y) = 0 \Rightarrow x = 0 \text{ OR } y = 0 \text{ OR } x = -y$$

So the solutions are (after eliminating repetitions)

$$\{[0, 0, 1], [0, 1, 0], [0, 1, -1], [1, 0, 0], [1, 0, -1], [1, -1, 0].\}$$

These points are the pairwise intersections of the four lines. (Note $\binom{4}{2} = 6$.) In particular, $\mathbb{V}(f_x, f_y, f_z)$ is a subset of $\mathbb{V}(xyz(x + y + z))$. And indeed, since Euler's theorem gives, in this case, that

$$x \cdot f_x + y \cdot f_y + z \cdot f_z = 4 \cdot f,$$

if $P \in \mathbb{V}(f_x, f_y, f_z)$ then f_x, f_y, f_z all vanish at P, so Euler's theorem implies that f vanishes at P, so in particular $P \in \mathbb{V}(f)$. □

FYI: The vanishing locus in $\mathbb{P}^2$ of a polynomial f that is a product of homogeneous linear polynomials, where none is a scalar multiple of another, is called a *line arrangement* and is an object of interest in current research. The vanishing locus of the partial derivatives, and the ideal that the partial derivatives generate, is an important part of that.

Exercise 23.

(a) $I = \langle x^4, y^5, z^6, x^2y^2z^3, x^3yz^4 \rangle$.

(b) $m_1 = 4$ since $x^4 \in I$. $m_2 = 5$ since $y^5 \in I$. $m_3 = 6$ since $z^6 \in I$. Thanks to the proof in class, we can take $r = 4 + 5 + 6 = 15$. But in fact $r = 13$ works. □

(c) $I = \langle x^2, y^4 \rangle$ and $J = \langle x^2 + y^4 \rangle$.

Exercise 24.
Let
$$P = [p_1, p_2, p_3], \quad Q = [q_1, q_2, q_3], \quad R = [r_1, r_2, r_3].$$
The fact that P, Q, R are collinear means that there is some linear form
$$ax + by + cz = [a\ b\ c\ ] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
that vanishes on all three points. (We're slightly abusing notation by identifying a 1×1 matrix with its entry.) That is, we have the matrix products
$$[a\ b\ c] \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0, \quad [a\ b\ c] \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0, \quad [a\ b\ c] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 0.$$
Then
$$[a\ b\ c]A^{-1}A \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0 \tag{13.4}$$
as well (and similarly for Q, R). Now, $[a\ b\ c]A^{-1}$ is a new 1×3 matrix of scalars, and as such it defines a new linear form
$$[a\ b\ c]A^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$
On the other hand,
$$A \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \phi(P). \tag{13.5}$$
Since (13.4) and (13.5) hold for Q and R as well, the equation (13.4) means that this new linear form vanishes on $\phi(P), \phi(Q), \phi(R)$ and so they are collinear.

The converse clearly holds since ϕ is invertible. □

Exercise 25.

(a) $I_\Lambda = \langle L_1, L_2 \rangle$, where L_1, L_2 are homogeneous linear polynomials in five variables.

(b) We want to find the common vanishing locus of two homogeneous linear polynomials, say $L_1 = a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3$ and $L_2 = b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3$. So we have a system of linear equations
$$\begin{array}{ccccccccc} a_0x_0 & + & a_1x_1 & + & a_2x_2 & + & a_3x_3 & = & 0 \\ b_0x_0 & + & b_1x_1 & + & b_2x_2 & + & b_3x_3 & = & 0. \end{array}$$
Consider the coefficient matrix
$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}.$$
The fact that the planes are distinct means that L_1 is not a scalar multiple of L_2, so the rows of A are independent. Thus the dimension of the solution space of this

system of equations is $4-2=2$ (where 4 is the number of variables and 2 is the number of equations). But a vector space of dimension two corresponds to a projective *line*, so we are done.

(c) Λ_1 could be defined by $\langle x_1-x_0, x_3-x_2\rangle$ and Λ_2 could be defined by $\langle x_2-x_0, x_4-x_3\rangle$. So $\Lambda_1 \cap \Lambda_2$ is the solution space of the system of equations

$$\begin{array}{rcl} x_1 - x_0 &=& 0 \\ x_3 - x_2 &=& 0 \\ x_2 - x_0 &=& 0 \\ x_4 - x_3 &=& 0 \end{array}$$

This means that if you fix any value for x_4, say $x_4 = \lambda$, then

$$\lambda = x_4 = x_3 = x_2 = x_0 = x_1,$$

so the solution is exactly the point $[1,1,1,1,1]$.

(d) There are infinitely many possible answers. For a linear form $L = a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + x_4x_4$ to vanish at the point $[1,1,1,1,1]$, we need

$$a_0 + a_1 + a_2 + a_3 + a_4 = 0$$

(plug the value 1 into each x_i). There is a four-dimensional linear space of such solutions. To get a plane we need to choose two independent ones (by a)), and there are infinitely many ways we could do that twice (to get two planes). □

Exercise 26.

(a) $\Leftarrow$:

If we know in advance that $a = 3t$, $b = 4t$ and $c = 5t$ then

$$ax + by + cz = 0 \quad \Leftrightarrow \quad (3t)x + (4t)y + (5t)z = 0 \quad \Leftrightarrow \quad 3x + 4y + 5z = 0$$

so they define the same line.

$\Rightarrow$:

Consider the lines $\mathbb{V}(ax + by + cz)$ and $\mathbb{V}(3x + 4y + 5z)$ in $\mathbb{P}^2$. Either they meet in a single point or they are the same line. To find out which, we solve a system of homogeneous linear equations

$$\begin{array}{ccccccc} 3x &+& 4y &+& 5z &=& 0 \\ ax &+& by &+& cz &=& 0. \end{array}$$

Each equation represents a plane through the origin in $\mathbb{R}^3$. The lines in $\mathbb{P}^2$ meet in a single point if and only if the solution space of these two equations is a 1-dimensional subspace of $\mathbb{R}^3$ (i.e. a line through the origin in $\mathbb{R}^3$, i.e. a point of $\mathbb{P}^2$). Looking at the coefficient matrix

$$\begin{bmatrix} 3 & 4 & 5 \\ a & b & c \end{bmatrix}$$

we know that the solution space is 1-dimensional if and only if the rank of this matrix is 2, if and only if neither row is a multiple of the other. So the lines are the same in $\mathbb{P}^2$ if and only if the solution space is 2-dimensional, if and only if $a = 3t$, $b = 4t$ and $c = 5t$ for some non-zero t as claimed.

(b) We have

$$(\mathbb{P}^2)^\vee = \{ \text{ Lines in } \mathbb{P}^2 \ \} = \{ \ \mathbb{V}(ax+by+cz) \ \} = \{ \ [a,b,c] \ \}$$

where the latter is the set of triples of real numbers, not all zero, up to scalar multiples, i.e. the latter is a projective plane.

$(\mathbb{P}^2)^\vee$ is called the **dual projective plane**. So what we have so far is that a point $P = [a,b,c]$ in $(\mathbb{P}^2)^\vee$ corresponds to the line $\ell_P = \mathbb{V}(ax+by+cz)$ in $\mathbb{P}^2$. You can use this for the next two parts even if you didn't get a) and/or b). Furthermore, even if you don't get c) you can use the statement of c) to do d) and e).

(c) Say $P_i = [a_i, b_i, c_i]$ for $i = 1,2,3$. Then the P_i all lie on a line in $(\mathbb{P}^2)^\vee$ if and only if there are some **constants** $p,q,r \in \mathbb{R}$ such that $[a_1,b_1,c_1],[a_2,b_2,c_2]$ and $[a_3,b_3,c_3]$ are all solutions to the equation

$$pa + qb + rc = 0$$

in the variables a,b,c. That is, we have

$$\begin{array}{ccccccc} a_1p & + & b_1q & + & c_1r & = & 0 \\ a_2p & + & b_2q & + & c_2r & = & 0 \\ a_3p & + & b_3q & + & c_3r & = & 0 \end{array}$$

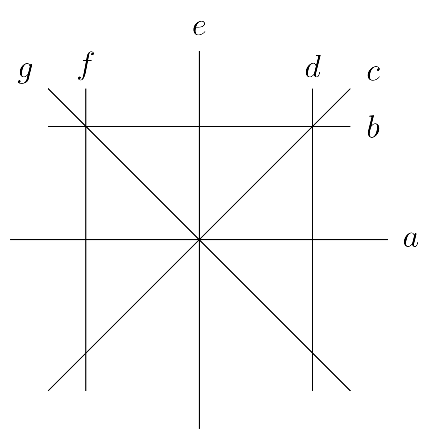
But this means that $[p,q,r]$ is a common solution of the equations

$$\begin{array}{ccccccc} a_1x & + & b_1y & + & c_1z & = & 0 \\ a_2x & + & b_2y & + & c_2z & = & 0 \\ a_3x & + & b_3y & + & c_3z & = & 0 \end{array}$$

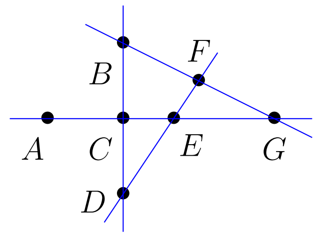
i.e. $[p,q,r]$ is common to the lines $\mathbb{V}(a_1x+b_1y+c_1z), \mathbb{V}(a_2x+b_2y+c_2z), \mathbb{V}(a_3x+b_3y+c_3z)$, i.e. to the lines $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ as desired.

(d) The points on this line are all on the same line (obviously), so the corresponding lines in $\mathbb{P}^2$ all pass through the same common point, by c). This collection of lines through a common point is called a **pencil** of lines.

(e) We start with the configuration



In sketching the dual set of points, we have to make sure that A, C, E, G are collinear, B, C, D are collinear, B, F, G are collinear and D, E, F are collinear. Here is one possible sketch. The blue lines are just to emphasize which points are collinear.



□

Exercise 27.
Let

$$f \in I^{sat} = \{f \in R \mid \text{for each } 0 \le i \le n \text{ there is some } m_i \text{ so that } x_i^{m_i} f \in I\}.$$

Write f as a sum of its homogeneous parts:

$$f = f_0 + f_1 + f_2 + \cdots + f_d.$$

We want to show that for each j, $f_j \in I^{sat}$. That is, having chosen f_j, we want to show that for each i we have $x_i^{m_i} f_j \in I$ for suitable m_i. Since $f \in I^{sat}$, we know that for each i we have $x_i^{m_i} f \in I$. But

$$x_i^{m_i} f = x_i^{m_i} f_0 + x_i^{m_i} f_1 + \cdots + x_i^{m_i} f_d$$

is the decomposition of $x_i^{m_i} f$ into its homogeneous parts. Since I is homogeneous, we have $x_i^{m_i} f_j \in I$ for each j, as desired. □

Exercise 28.

(a) For $d \ge 4$, notice that every monomial is in $\langle x^2, y^2, z^2 \rangle$, i.e. $[\langle x^2, y^2, z^2 \rangle]_j = [R]_j$ for all $j \ge 4$. (Soon we will give this property a name: it is an *artinian* ideal.) So $1 \in \langle x^2, y^2, z^2 \rangle^{sat}$, i.e. the saturation is all of $k[x, y, z]$.

(b) Let $I = \langle x^2, y^2, z^2 \rangle$. Now it is no longer true that $[I]_j = [\langle x^2, y^2, z^2 \rangle]_j = [R]_j$ for any $j \ge 0$. (For example, w^j is never in I.) In fact, we claim that I is already saturated! (See what a difference an extra variable can make? Compare with (a).)

We know that $I \subset I^{sat}$ is always true, so we want to prove the reverse inclusion. Let $f \in I^{sat}$, so there exist m_0, m_1, m_2, m_3 such that $f w^{m_0} \in I, f x^{m_1} \in I, f y^{m_2} \in I, f z^{m_3} \in I$. Ignoring the last three, consider the condition $f w^{m_0} \in I$. We have

$$f w^{m_0} = Ax^2 + By^2 + Cz^2. \tag{13.6}$$

By unique factorization, w^{m_0} has to divide $Ax^2 + By^2 + Cz^2$. We can't quite conclude that w^{m_0} divides each of A, B and C since for instance we might have $A = -y^2$ and $B = x^2$, in which case we only conclude that w^{m_0} divides C. So assume that no single term in the right-hand side of (13.6) is zero (i.e. $A \neq 0, B \neq 0, C \neq 0$), that no two terms sum to zero, and in fact that if we expand all products, we have removed any terms that cancel out. This means that w^{m_0} divides every term on the right. Since

w^{m_0} clearly has no factor in common with x^2, y^2 or z^2, this means that it divides A, B and C. Then dividing both sides by w^{m_0}, and we get

$$f = A'x^2 + B'y^2 + C'z^2 \in I.$$

Thus $I = I^{sat}$ and we are done.

(c) Notice that $[\langle x^2, xy, xz\rangle]_j = [\langle x\rangle]_j$ for all $j \geq 2$, so the desired saturation is $\langle x\rangle$. □

Exercise 29.

Assume $V \subset \mathbb{P}^n$ is a projective variety and let $R = k[x_0, x_1, \ldots, x_n]$. We know that $I_V \subseteq I_V^{sat}$, so we want to show the opposite inclusion. Let $f \in I_V^{sat}$. We want to show that $f \in I_V$, i.e. that $f(P) = 0$ for all $P \in V$. Let $P \in V$. We know that $\mathbb{V}(x_0, x_1, \ldots, x_n) = \emptyset$, so there is at least one x_i that does not vanish at P. But for this choice of x_i we still have $f \cdot x_i^{m_i} \in I_V$ for some m_i, so it vanishes at P. Since $x_i^{m_i}$ does not vanish at P, we must have $f(P) = 0$ as desired. □

Exercise 30.

We have $I = \langle x^2, xy, xz\rangle \subset R = k[x, y, z]$. Notice that $I = x \cdot \langle x, y, z\rangle$, i.e. the generators of I are generators of the degree 2 component of the ideal $\langle x\rangle$. So $[I]_t = [\langle x\rangle]_t$ for all $t \geq 2$, so the Hilbert functions coincide. We get $h_{R/I}(t) = t+1$ for all $t \geq 2$. Since $R/\langle x\rangle$ has depth 1 and R/I agrees with $R/\langle x\rangle$ in all degrees ≥ 2, there is no other degree where $\times L$ fails to be injective. The saturation of I is $\langle x\rangle$, and it corresponds to a line in $\mathbb{P}^2$. □

Exercise 31.

(a) We saw in Exercise 29 that I_V is a saturated ideal. The condition that $L\bar{G} = 0$ in R/I_V means that $LG \in I_V$. It's easy to see that $L \notin I_V$ and in fact L does not vanish on either component of V. Thus if LG vanishes on all of V, we must have $G \in I_V$. This means $\bar{G} = 0$ in R/I_V. So L is a regular element by definition.

(b) The plane defined by L meets the component $\mathbb{V}(x_0, x_1)$ at the point

$$\mathbb{V}(x_0, x_1, x_0 + x_1 + x_2 + x_3) = \mathbb{V}(x_0, x_1, x_2 + x_3) = [0, 0, 1, -1].$$

Similarly, the plane defined by L meets the component $\mathbb{V}(x_2, x_3)$ at the point

$$\mathbb{V}(x_2, x_3, x_0 + x_1 + x_2 + x_3) = \mathbb{V}(x_2, x_3, x_0 + x_1) = [1, -1, 0, 0].$$

(c) By inspection we can choose $L' = x_0 + x_1$.

(d) We check each x_i separately.

$$\begin{aligned}
x_0(x_0 + x_1) &= x_0^2 + x_0x_1 = x_0(x_0 + x_1 + x_2 + x_3) - x_0x_2 - x_0x_3 \in \langle L, I_V\rangle \\
x_1(x_0 + x_1) &= x_0x_1 + x_1^2 = x_1(x_0 + x_1 + x_2 + x_3) - x_1x_2 - x_1x_3 \in \langle L, I_V\rangle \\
x_2(x_0 + x_1) &= x_0x_2 + x_1x_2 \in I_V \subset \langle L, I_V\rangle \\
x_3(x_0 + x_1) &= x_0x_3 + x_1x_3 \in I_V \subset \langle L, I_V\rangle
\end{aligned}$$

(e) No matter what element of $[R/\langle L, I_V\rangle]_1$ you choose, part (d) shows that it is annihilated by $x_0 + x_1$. Notice that $(x_0 + x_1) \neq 0$ in $R/\langle L, I_V\rangle$. So for a general linear form ℓ, the equation $\ell G = 0$ always has a nonzero solution, namely $G = x_0 + x_1$. Thanks to Remark 3.8, this means that $R/\langle L, I_V\rangle$ has no non-zerodivisors, and $\operatorname{depth}(R/I_V) = 1$.

(f) $x_0, x_2 \neq 0$ in R/I_V but $x_0 \cdot x_2 = 0$ in R/I_V. □

Exercise 32.

We know that $I \subseteq I^{sat}$, so $[I]_t \subseteq [I^{sat}]_t$ for all $t \geq 0$. The exercise is asserting that the number of degrees in which this latter is not an equality is finite. For convenience denote by $\mathfrak{m}$ the irrelevant ideal $\langle x_0, \dots, x_n \rangle$.

Since $R = k[x_0, \dots, x_n]$ is Noetherian, I^{sat} is finitely generated. Let d be the largest degree of any element in a minimal generating set for I^{sat}. Let $\{f_1, \dots, f_r\}$ be a basis for $[I^{sat}]_d$. (These elements may or may not be in I.) For each f_i and each variable x_j, $0 \leq j \leq n$, there is a positive integer $m_{i,j}$ so that $f_i \cdot x_j^{m_{i,j}} \in I$. It's not hard to check that then for each i there exists a positive integer N_i (for example the sum over j of the $m_{i,j}$ works) so that $f_i \cdot \mathfrak{m}^{N_i+p} \subset I$ for all $p \geq 0$.

Now let $N = \max_i\{N_i\}$, so that $f_i \cdot \mathfrak{m}^N \subset I$ for all f_i. It follows that $[I^{sat}]_d \cdot \mathfrak{m}^N \subseteq [I]_{d+N}$. Since all the minimal generators of I^{sat} occur in degree $\leq d$, we know that $\mathfrak{m}^p \cdot [I^{sat}]_d = [I^{sat}]_{d+p}$ for any $p \geq 0$. Putting it all together we have

$$[I]_{d+N+p} \subseteq [I^{sat}]_{d+N+p} = \mathfrak{m}^{N+p} \cdot [I^{sat}]_d \subseteq [I]_{d+N+p}$$

(since the f_i generate I^{sat}). This gives the result. □

Exercise 33.

(a) Let $f \in I : \mathfrak{m}$ and write f as the sum of its homogeneous parts:

$$f = f_0 + f_1 + \cdots + f_d.$$

We want to show that each f_i is in $I : \mathfrak{m}$. Let $m \in \mathfrak{m}$. Without loss of generality assume m is homogeneous. (If not, apply the same argument for each homogeneous part.) By definition, $fm \in I$. Note that

$$fm = f_0 m + f_1 m + \cdots + f_d m.$$

This is the homogeneous decomposition of fm since m is a homogeneous polynomial. Since I is a homogeneous ideal and $fm \in I$, each $f_i m \in I$. But this means that each f_i is in $I : \mathfrak{m}$, as desired.

(b) It's clear that $I : \mathfrak{m} \supset I$ always, so really we can replace $I : \mathfrak{m} = I$ with $I : \mathfrak{m} \subseteq I$ in the statement.

Assume first that I is saturated. Let $f \in I : \mathfrak{m}$. We want to show that $f \in I$. Since $f \in I : \mathfrak{m}$, we have $fx_0 \in I$, $fx_1 \in I$, $\dots$, $fx_n \in I$. By Definition 3.3, this means $f \in I^{sat}$. But $I^{sat} = I$ since I is saturated, so we are done.

Conversely, assume that $I : \mathfrak{m} \subseteq I$. We want to show that $I = I^{sat}$. Since I^{sat} in any case contains I, we can suppose that I^{sat} properly contains I and seek a contradiction. We have seen in Exercise 32 that for $t \gg 0$ we have $[I^{sat}]_t = [I]_t$, so it makes sense to choose f homogeneous of largest possible degree so that $f \in I^{sat} \backslash I$. We claim that then we have

$$fx_0, \dots, fx_n \in I. \tag{13.7}$$

Certainly since $f \in I^{sat}$, some power of each x_i multiplies f into I, so if $fx_i \notin I$ for some i we can replace f by fx_i, contradicting the assumption that f is of largest possible degree. But (13.7) implies that $f \in I : \mathfrak{m} = I$, so we are done.

(c) This is essentially what we proved in (b) via (13.7). □

Exercise 34

Suppose that I is not saturated. We want to show that $\operatorname{depth}(R/I) = 0$. By Exercise 33 (c), the fact that I is not saturated means that R/I has a socle element f, so for any (linear) form L the equation $LG = 0$ in R/I does not force $G = 0$, since we can always take $G = f$ no matter what L is. So R/I has no non-zerodivisors, i.e. $\operatorname{depth}(R/I) = 0$. □

Exercise 35.

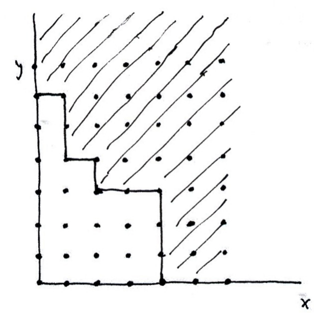
(a) From Example 3.16 we have seen that $\operatorname{Kdim}(R) = n+1$, while it is not hard to show that $(x_0, \dots, x_n)$ is a regular sequence.

(b) Since V is a finite union of points, the Krull dimension of R/I_V is 1. On the other hand, if L is a linear form defining a hyperplane that avoids all the points of V then it is a non-zerodivisor since $LF \in I_V$ forces $F \in I_V$.

(c) Take $C =$ two skew lines. It is a union of two copies of $\mathbb{P}^1$ so it is a variety of dimension 1, and hence $\operatorname{Kdim} R/I_C = 2$, while in Exercise 31 showed that $\operatorname{depth}(R/I_C) = 1$. □

Exercise 36.

(a) In the following picture, the dots represent monomials.



(b)

$$1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, y^4, x^3y^2, y^5.$$

(c) Count the number of dots on the diagonals, not in the shaded area.

$$h_{R/I}(t) = \begin{cases} 1 & \text{if } t = 0; \\ 2 & \text{if } t = 1; \\ 3 & \text{if } t = 2; \\ 4 & \text{if } t = 3; \\ 4 & \text{if } t = 4; \\ 2 & \text{if } t = 5; \\ 0 & \text{if } t \geq 6; \end{cases}$$

(d) The Hilbert polynomial of R/I is the zero polynomial, since R/I takes the value 0 for all $t \geq 6$. □

Exercise 37
The given sequence of integers is $(1, 5, 12, 17, 25, 36)$. The growth from degree 0 to degree 1 is automatically OK, and the growth from degree 5 on is automatically OK. For the rest:

$$5 = \binom{5}{1} \Rightarrow 5^{(1)} = \binom{6}{2} = 15.$$

$$12 = \binom{5}{2} + \binom{2}{1} \Rightarrow 12^{(2)} = \binom{6}{3} + \binom{3}{2} = 20 + 3 = 23.$$

$$17 = \binom{5}{3} + \binom{4}{2} + \binom{1}{1} \Rightarrow 17^{(3)} = \binom{6}{4} + \binom{5}{3} + \binom{2}{2} = 15 + 10 + 1 = 26.$$

$$25 = \binom{6}{4} + \binom{5}{3} \Rightarrow 25^{(4)} = \binom{7}{5} + \binom{6}{4} = 21 + 15 = 36.$$

Since

$$12 < 15,\ 17 < 23,\ 25 < 26 \ \text{ and } \ 36 \leq 36,$$

the sequence is an O-sequence. Notice that the growth from degree 4 to degree 5 is maximal. (So if the sequence had ended with 37 instead of 36, it would not be an O-sequence.)

Exercise 38
By the Auslander-Buchsbaum formula,

$$\text{proj dim } R/I + \text{depth } R/I = n + 1 = 2.$$

Since R/I is artinian, its depth is 0. Thus the projective dimension is 2. We also know that it is Gorenstein. Thus the minimal free resolution has the form

$$0 \to R(-) \to \mathbb{F} \to R \to R/I \to 0.$$

But the alternating sum of the ranks is 0, so $\mathbb{F}$ has to have rank 2. This is the codimension of R/I so R/I is a complete intersection.

Exercise 39
See the instructors if you need help or suggestions.

Exercise 40
We saw in Example 3.19 that a single line is ACM so we just have to show that a set V of two skew lines is not ACM.

A set of two skew lines (in any projective space) has Krull dimension 2, since the two skew lines are one-dimensional as a projective variety. On the other hand, since by definition I_V is saturated, we have by Remark 3.11 that R/I has depth at least one.

So to show that V is not ACM, we have to show that the depth of R/I_V is exactly 1. That is, there does not exist a regular sequence of length 2. By Remark 3.8, it is enough to look at linear forms. Then we are done by Exercise 31.

Exercise 41 Let $Z \subset \mathbb{P}^n$ be a set of d points and denote by $h_Z(t)$ its Hilbert function. It is trivially true that $h_Z(t) = 0$ for $t \leq -1$ and $h_Z(0) = 1$. If $n = 0$ then Z is a single point and $h_Z(t) = 1$ for all $t \geq 1$ so there is nothing to prove. Thus we assume $n \geq 1$. If $d = 1$, we have

$h_Z(t) = 1$ for all $t \geq 1$ and again there is nothing to prove. So without loss of generality, assume $d \geq 2$; then we also have $h_Z(1) > h_Z(0)$.

At this point, without loss of generality we can assume $t \geq 2$ and $d \geq 2$.

Let I_Z be the defining homogeneous ideal of Z. Let L be a linear form not vanishing on any of the points of Z. We first claim that $I_Z : L = I_Z$. Indeed, if $LF \in I_Z$ then $F \in I_Z$ since L avoids all the points, so the claim follows immediately.

Then the exact sequence in Remark 3.12 gives us a short exact sequence

$$0 \to [R/I_Z]_{t-1} \xrightarrow{\times L} [R/I_Z]_t \to [R/\langle I, L\rangle]_t \to 0.$$

Thus we get $h_Z(t-1) \leq h_Z(t)$ for all t. It only remains to show that once $h_Z(t_0-1) = h_Z(t_0)$ for some t_0, we have equality for all $t \geq t_0$. But the stated equality means

$$[R/\langle I, L\rangle]_{t_0} = 0.$$

Since $R/\langle I, L\rangle$ is a standard graded algebra, as an R-module it is generated in degree 0. Thus once a component is zero, it can never become non-zero. So the Hilbert function is strictly increasing from degree 0 until some t_0, at which point it stabilizes.

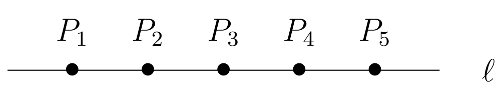
Why is the value of the Hilbert function at this point precisely d? As in Example 7.3 (c), you can check that for $t \gg 0$, Z imposes independent conditions on forms of degree t. For such t,

$$h_Z(t) = \dim[R/I_Z]_t = \dim[R]_t - \dim[I_Z]_t = \dim[R]_t - (\dim[R]_t - d) = d.$$

Exercise 42.

(a) Let $P_1, \ldots, P_5$ be a set of five points in $\mathbb{P}^2$. We'll prove that if they lie on a line then they do not impose independent conditions on cubics, and if they do not lie on a line then they do impose independent conditions on cubics.

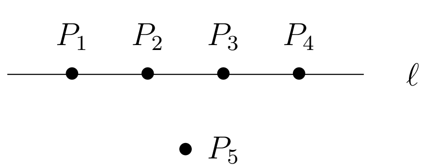
Assume that $P_1, \ldots, P_5$ lie on a line, ℓ.



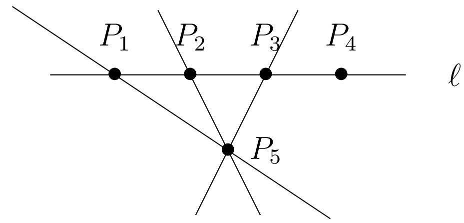
We want to know: if we remove any point, say P_i, can we find a cubic vanishing at all the remaining points but not at P_i? Say F were such a cubic. Then the restriction of F to the line $\ell \cong \mathbb{P}^1$ would be a homogeneous polynomial of degree 3 with four zeros. But then this restriction has to be identically zero. This means that F vanishes along all of ℓ, so in particular it vanishes at P_i. Thus the points do not impose independent conditions on forms of degree 3, i.e. on plane cubics.

Now assume that the points do not all lie on a line.

Case 1: Four of the points are on a line, say ℓ, and the fifth is not on that line. Without loss of generality say P_5 is not on the line.

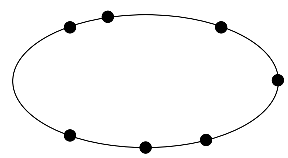


We want to remove any of the five points and show that there is a cubic vanishing at the remaining points but not the one we removed. If we remove P_5, for example the cubic ℓ^3 does the trick. If we remove any of the other points, without loss of generality say it is P_4 (but the same argument works for any of the points on ℓ). Then the cubic consisting of the union of the three lines joining P_5 to P_1, P_2, P_3 respectively does the trick:



Case 2: Assume no four of the points lie on a line. In this case we can subdivide into the subcase where three of the points lie on a line, and the subcase where no three lie on a line. In both subcases, though, it's easy to see that you can use three lines to isolate any of the five points, as we did above.

(b)



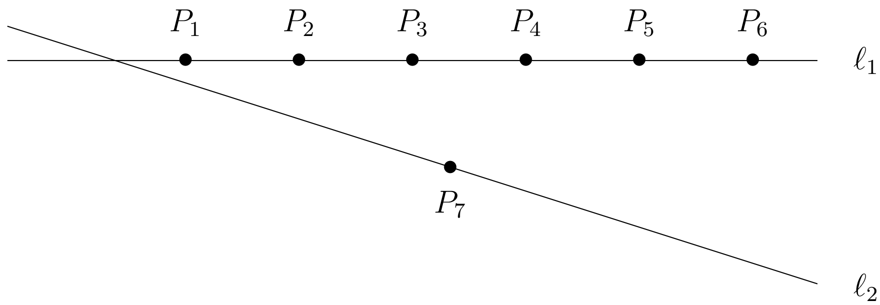
Clearly not all the seven points lie on a line, so $\dim[I(V)]_0 = \dim[I(V)]_1 = 0$. This accounts for the 1 and the 3. Also, clearly once we reach the value 7 it stays at 7, from what we said in class. So we have to verify the values in degrees 2 and 3. It is a fact (e.g. from [H]) that two conics contain at most four points in common, unless they have a common factor. Since our conic is irreducible, it does not contain a common factor with anything else. Thus we can't have two independent conics containing V, so $\dim[I(V)]_2 = 1$ and so

$$h_V(2) = \dim[R]_2 - \dim[I(V)]_2 = 6 - 1 = 5.$$

Finally, to verify the value in degree 3 it's enough to show that the points impose independent conditions on cubics. But removing any point P_i, we can pair the remaining points up and consider the three lines that we thus get. Since each of these lines already contains two points of the conic, the hint shows that they can't contain a third, i.e. P_i is not on the cubic formed by the union of these three lines. This completes the proof.

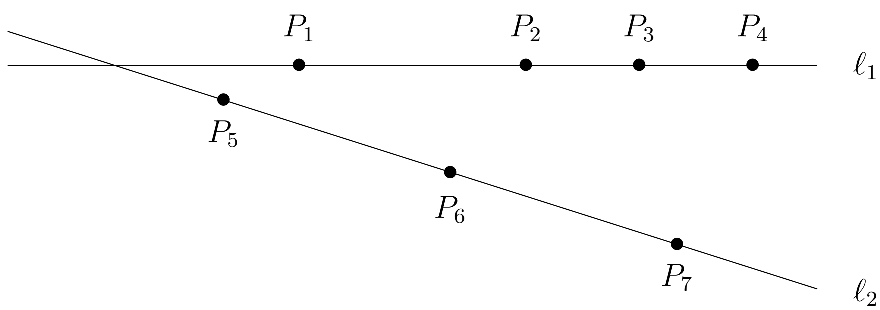
(c) The 3 says that the points lie in $\mathbb{P}^2$ but do not all lie on a line (otherwise it would be $3 - 1 = 2$). As above, the 5 says that the points lie on a **unique** conic. But if they lay on an irreducible conic, we would get the Hilbert function from (b), which is not the Hilbert function we're looking at. So these points lie on a reducible conic, i.e. a union of two lines. The 7's tell us that we have a total of seven points. So the key is to see what the 6 tells us.

The issue is to see how many points lie on one line and how many lie on the other. We just saw that not all seven lie on one of them. Suppose that six lie on one line and one lies on the other:



Then we could replace ℓ_2 by any other line containing P_7, so there would not be a unique conic containing the seven points. So this is impossible.

We're left with either 4 on one line and 3 on the other, or 5 on one line and 2 on the other. Let's rule out the former.



If we remove one of P_1, P_2, P_3, P_4 then the remaining six can be paired up with lines joining a point on ℓ_1 with a point on ℓ_2, avoiding the one we removed. If we remove one of P_5, P_6, P_7, say without loss of generality it's P_5, then we take ℓ_1 together with a line other than ℓ_2 through P_6 and one through P_7. We conclude from this that this set of seven points imposes independent conditions on cubics, so the value of the Hilbert function in degree 3 is 7, not 6.

The only remaining possibility is that we have five points on one line (say ℓ_1) and two on the other (or else that this Hilbert function does not occur at all, but this is not the case). Let's verify that a set of points of this sort **does** have the desired Hilbert function. The same kind of reasoning as in part (a) shows that this set of points does not impose independent conditions on cubics, but it does impose independent conditions on quartics. Remember that a finite set of points imposes independent conditions on forms of degree t if and only if the value of the Hilbert function in degree t is the number of points. So we have the following information:

t	$h(t)$	
0	1	
1	3	(since the points don't all lie on a line)
2	5	(since the points lie on a unique conic)
3	?	(but not yet 7)
4	7	
≥ 5	7	

We said in class that the Hilbert function has to be strictly increasing until you reach the number of points. Therefore the value in degree 3 has to be 6, and we're done. □

Exercise 43.

(a) I_C is a monomial ideal, so as before we can count monomials not in I_C.

t	basis for $[R/I]_t$	$h_{R/I}(t)$	
0	1	1	
1	w, x, y, z	4	
2	$w^2, wx, x^2, y^2, yz, z^2$	6	(Note $6 = 10 - 4$.)
3	$w^3, w^2x, wx^2, x^3, y^3, y^2z, yz^2, z^3$	8	
$\vdots$			
t	$w^t, w^{t-1}x, w^{t-2}x^2, \dots, wx^{t-1}, x^t$		
	$y^t, y^{t-1}z, y^{t-2}z^2, \dots, yz^{t-1}, z^t$	$2t+2$	

(b)

$$\begin{array}{ll} h_{R/I_C} & 1, 4, 6, 8, \dots \\ \Delta h_{R/I_C} & 1, 3, 2, 2, \dots \\ \Delta^2 h_{R/I_C} & 1, 2, -1, 0, \dots \end{array}$$

Notice that the entries of $\Delta^2 h_{R/I_C}$ are not all positive. This is related to the fact that you proved in Exercise 31 that $\operatorname{depth}(R/I_C) = 1$, while $\operatorname{Kdim}(R/I_C) = 2$, so C is not ACM. □

Exercise 44.

Remark 4.11 says that if V is a finite set of points then the eventual value of $h_V(t)$ is the number of points, and Remark 7.4 shows that if R/I is Cohen-Macaulay (e.g. if $I = I_V$ for some finite set of points V) then you recover h_{R/I_V} by "integrating." Thus the Hilbert function in this case is given by

$$(1, 1 + a_1, 1 + a_1 + a_2, \dots)$$

and the eventual value is $1 + a_1 + a_2 + \cdots + a_d$ as claimed. □

Exercise 45.

The artinian reduction of R/I has Hilbert function equal to the appropriate difference of the original Hilbert function exactly when R/I is Cohen-Macaulay. This is because we need (7.1) to be a short exact sequence, and for this we need a regular sequence of the right length. □

Exercise 46.

The degree is obtained by adding the entries of the h-vector, namely 22.

For the Hilbert function, it's the same idea as before: we integrate.

dimension	Hilbert function
artinian	$(1, 4, 7, 8, 2)$
points	$1, 5, 12, 20, 22, 22, \dots$
curve	$1, 6, 18, 38, 60, 82, \dots$
surface	$1, 7, 25, 63, 123, 205, \dots$

□

Exercise 47.

If I were saturated, R/I would have depth at least 1 (Exercise 34). Thus for a general linear form L, the Hilbert function of $R/\langle I, L\rangle$ would be $(1, 3, -1, 1, 1, \dots)$. This is clearly not the Hilbert function of any standard graded algebra. Thus I can't be saturated.

The Hilbert polynomial of R/I is clearly $t + 1$, so the leading coefficient tells us (Remark 4.7) that I is eventually equal to the homogeneous ideal of a line, which has Hilbert function $(1, 2, 3, \dots)$ (Example 7.3 (4)). So I^{sat} is the ideal of a line, and its Hilbert function is $(1, 2, 3, \dots)$. □

Exercise 48.

We'll use Lemma 8.4. Let L be a general linear form and consider the multiplication by L on R/I from degree $t-1$ to degree t. Assume that R/I has the WLP. Consider the exact sequences

$$0 \to \left[\frac{I : L}{I}\right]_{t-1} \to \left[\frac{R}{I}\right]_{t-1} \xrightarrow{\times L} \left[\frac{R}{I}\right]_t \to \left[\frac{R}{\langle I, L\rangle}\right]_t \to 0$$

and

$$0 \to [R/(I : L)]_{t-1} \xrightarrow{\times L} [R/I]_t \to R/\langle I, L\rangle \to 0$$

(see (3.1)).

Surjectivity of $\times L$ is equivalent to $[R/\langle I, L\rangle]_t = 0$. It is clear that once this is zero for some t, it is zero forever after that (since once $\langle I, L\rangle$ is equal to R in one degree, it is equal forever after). In other words, once you have surjectivity in one degree, it is surjective forever after.

But WLP means that if surjectivity does not hold then injectivity does. Injectivity implies $h_{R/I}(t-1) \leq h_{R/I}(t)$, and equality means that $\times L$ is both injective and surjective. Surjectivity means $h_{R/I}(t-1) \geq h_{R/I}(t)$.

Clearly $\times L$ is injective but not surjective when $t = 0$. Then $\times L$ must be injective but not surjective for a while (possibly) – this corresponds to $h_{R/I}$ being strictly increasing – then possibly both injective and surjective, and then surjective. The result follows.

One caveat is that it is possible that the tail of the Hilbert function does have places where the values are equal. For example, $(1, 3, 6, 8, 8, 6, 6, 4, 3, 3, 1)$ is possible. Here we have only injectivity until $t = 3$, then surjectivity for $t \geq 4$, but in fact we have both injectivity and surjectivity for $t = 4, 6, 9$ (and of course after the Hilbert function reaches 0).

Exercise 49.

We want to show that if R/I is a monomial algebra (i.e. I is generated by monomials) then R/I has the WLP if and only if multiplication by $L = x_1 + \cdots + x_n$ has maximal rank in each degree, where $R = k[x_1, \dots, x_n]$.

We already know that if this L gives maximal rank then it also holds for a general element of $[R]_1$, by semicontinuity. So we want to prove the converse. That is, assume that we know that R/I has the WLP, so there is some element L' for which $\times L'$ has maximal rank in each degree.

Claim 1: Since R/I is artinian, some power of each variable is a minimal generator of I. (You should convince yourself of this.)

Claim 2: Recall that since, by assumption, L' gives maximal rank in each degree, it is also true for a general element of $[R]_1$. So without loss of generality we can assume that $L' = a_1x_1 + \cdots + a_nx_n$ with $a_i \neq 0$ for all $1 \leq i \leq n$.

Claim 3: Performing a change of variables does not change whether or not R/I has WLP. So use the substitution

$$x_i \mapsto \frac{1}{a_i}x_i.$$

Claim 4: Under this change of variables, $L \mapsto x_1 + \cdots + x_n$.

Claim 5: Nevertheless, the monomial ideal I remains unchanged.

This completes the proof. Also see the attached paper [MMN1] (Proposition 2.2) and the class notes.

Exercise 50.

(a) For a monomial ideal I, a basis for $[R/I]_t$ is given by all the monomials of degree t not in I. In our case we have

t	basis for $[R/I]_t$
0	1
1	x, y, z
2	xy, xz, yz
3	xyz
$t \geq 4$	0

(b) A basis for $[R/I]_0$ is given by 1, and clearly $1 \cdot L = L \neq 0$ so injectivity is clear from degree 0 to degree 1. As for the multiplication from degree 2 to degree 3, we saw that a basis for $[R/I]_2$ is given by $\{xy, xz, yz\}$, and a basis for $[R/I]_3$ is given by $\{xyz\}$. Consider

$$(x + y + z)(axy + bxz + cyz) = (a + b + c)xyz$$

in R/I. For example taking $a = 1, b = c = 0$ gives surjectivity.

(c) Now consider the multiplication from degree 1 to degree 2. Using the given bases for $[R/I]_1$ and $[R/I]_2$ we get that $\times L$ is represented by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The determinant of this matrix is -2, which is zero if and only if k has characteristic 2. The conclusion about WLP is immediate.

(d) Using the given bases, an element $f = ax + by + cz$ of $[R/I]_1$ can be represented by the column matrix $[a\ b\ c]^t$, so the multiplication gives

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ a+c \\ b+c \end{bmatrix}.$$

Since we are in characteristic 2, taking $a = b = c = 1$ does the trick. Notice that this means L itself is in the kernel of $\times L$, i.e. that $L^2 = 0$ in R/I. ∎

Exercise 51.

(a) I contains a pure power of each variable, so it is artinian.

(b) As in Exercise 50, for a monomial ideal I, a basis for $[R/I]_t$ is given by all the monomials of degree t not in I. In our case we have

t	basis for $[R/I]_t$	$h_{R/I}(t)$
0	1	1
1	x, y, z	3
2	$x^2, xy, xz, y^2, yz, z^2$	6
3	$x^2y, x^2z, xy^2, xz^2, y^2z, yz^2$	6
4	x^2y^2, x^2z^2, y^2z^2	3
$t \geq 5$	0	

so the Hilbert function is $(1, 3, 6, 6, 3)$.

(c) We want to show that $\times(x + y + z)$ fails maximal rank, no matter what the characteristic of k is. Consider the exact sequence

$$[R/I]_2 \xrightarrow{\times L} [R/I]_3 \to [R/\langle I, L\rangle]_3 \to 0.$$

where $L = x + y + z$. It is enough to show that

$$\dim[R/\langle x^3, y^3, z^3, xyz, x + y + z\rangle]_3 > 0.$$

We have

$$\begin{aligned} k[x, y, z]/\langle x^3, y^3, z^3, xyz, x + y + z\rangle &\cong k[x, y]/\langle x^3, y^3, (x + y)^3, xy(x + y)\rangle \\ &\cong k[x, y]/\langle x^3, y^3, x^3 + 3x^2y + 3xy^2 + y^3, xy(x + y)\rangle \\ &\cong k[x, y]/\langle x^3, y^3, 3xy(x + y), xy(x + y)\rangle \\ &\cong k[x, y]/\langle x^3, y^3, xy(x + y)\rangle \end{aligned}$$

so this is clearly non-zero in degree 3 since $\dim[k[x, y]]_3 = 4$. At no point did the characteristic play a role in our calculation, so it is independent of the characteristic. □

Exercise 52.

We have assumed that R/I is an artinian Gorenstein algebra with the WLP, and that $\underline{h}$ is its Hilbert function. We want to show that $\underline{h}$ is an SI-sequence.

By Definition 10.1, we have to show that $\underline{h}$ is symmetric and that its first difference up to the middle is an O-sequence. The first of these is automatic since R/I is Gorenstein (see Remark 5.1) so we focus on the second one.

Recall the exact sequence from Remark 3.12:

$$0 \to \left[\frac{I : L}{I}\right]_{t-1} \to \left[\frac{R}{I}\right]_{t-1} \xrightarrow{\times L} \left[\frac{R}{I}\right]_t \to \left[\frac{R}{\langle I, L\rangle}\right]_t \to 0$$

We have the following facts.

1. The WLP tells us that the first vector space and the last vector space in this exact sequence can never be non-zero at the same time.
2. $R/\langle I, L\rangle$ is a standard graded algebra, so once it is zero in some degree, it is zero forever after. (This is observed in Lemma 8.4.)
3. By duality, we must have injectivity in the first half and surjectivity in the second half. (See also Proposition 11.1.)

This means that we have injectivity up to the middle, so Δh is the Hilbert function of $R/\langle I, L\rangle$ up to the middle, hence is an O-sequence.

Exercise 53.

This is immediate from the fact that

$$\binom{k}{n} - \binom{k-1}{n} = \binom{k-1}{n-1}$$

for any $k, n > 0$.

Exercise 54.

The following is from [MZ3] but you can come up with your own example. Consider the sequence

$$(1, 10, 14, 20, 14, 10, 1).$$

It is obviously symmetric and unimodal. It is an O-sequence because

$$14 \leq 10^{(1)} = 55, \quad 20 \leq 14^{(2)} = 30$$

(and the rest is immediate because it is non-increasing).

However, the first difference is $(1, 9, 4, 6)$, and

$$6 > 4^{(2)} = 5$$

so this is not an O-sequence.

It is shown in [MZ3] that this sequence is actually the Hilbert function of a suitable artinian Gorenstein algebra.

Exercise 55.

(a) The polynomial ring $R = k[x, y]$ satisfies

$$\dim[R]_i = \binom{i+2-1}{i} = i + 1.$$

Let $I = \langle f, g\rangle$, where $\deg f = m$ and $\deg g = n \geq m$. We have the Koszul resolution (see page 19)

$$0 \to R(-m-n) \to R(-m) \oplus R(-n) \to R \to R/I \to 0$$

which gives

$$\dim[R/I]_i = \left\{ \begin{array}{ll} i+1 & \text{for } 0 \leq i < m; \\ (i+1) - (i-m+1) & \text{for } m \leq i < n; \\ (i+1) - (i-m+1) - (i-n+1) & \text{for } n \leq i < m+n; \\ (i+1) - (i-m+1) - (i-n+1) + (i-m-n+1) & \text{for } i \geq m+n \end{array} \right.$$

$$= \left\{ \begin{array}{ll} i+1 & \text{for } 0 \leq i < m; \\ m & \text{for } m \leq i < n; \\ m+n-i-1 & \text{for } n \leq i < m+n; \\ 0 & \text{for } i \geq m+n. \end{array} \right\}$$

as desired.

(b) First, it is clearly symmetric.

Second, note that for any t,

$$t+1 = \binom{t+1}{t}$$

so

$$(t+1)^{(t)} = \binom{t+2}{t+1} = t+2.$$

Since the growth of the given sequence is never greater than this, it is an O-sequence. (This is obvious anyway since it is the Hilbert function of a specific algebra.)

Finally, the first difference of the first half of the sequence is the constant sequence $(1,1,\dots,1)$, which is clearly also an O-sequence.

Exercise 56.

Let $Z = \{P_1,\dots,P_r\}$. Let $\tau = \binom{t+n}{n}$, and note that $\dim[R]_t = \tau$. Let

$$f = a_1x_0^t + \cdots + a_\tau x_n^t \in [R]_t.$$

Then vanishing at any P_i gives a homogeneous linear equation in the variables $a_0,\dots,a_\tau$, and solving the r linear equations gives $[I \cap I_Z]_t$. Showing that these linear equations are independent is the same as showing that none of them is dependent on the other $r-1$, which boils down to showing that the removal of any of the points has a solution that does not vanish at the last one. □

Exercise 57.

For $P = [1,0,\dots,0]$ we have $I_{mP} = I_P^m = \langle x_1,\dots x_n\rangle^m$. Then

$$\dim_{\mathbb{C}}(\mathbb{C}[x_0,x_1,\dots,x_n] \cap [I_P^m]_t) = \dim_{\mathbb{C}}[I_P^m]_t.$$

Recall from Exercise 1 that $\dim_{\mathbb{C}}(\mathbb{C}[x_0,\dots,x_n]_t) = \binom{t+n}{n}$; hence we write

$$\dim_{\mathbb{C}}[I_P^m]_t = \binom{t+n}{n} - \left(\binom{t+n}{n} - \dim_{\mathbb{C}}[I_P^m]_t\right).$$

Therefore the number of conditions imposed by mP on forms of degree t is the number of monomials of degree t in $C[x_0,x_1,\dots,x_n]$ that are not in $(x_1,\dots x_n)^m$. This number is $\binom{t+n}{n}$ if $m > t$; otherwise it is

$$\sum_{j=0}^{m-1}\binom{n-1+j}{j} = \binom{n+m-1}{m-1} = \binom{n+m-1}{n},$$

where each summand is the number of monomials of type $x_0^{t-j}\cdot M$ with $M \in \mathbb{C}[x_1,\dots,x_n]_j$, and we are using the well-known Pascal's rule $\binom{d}{k} + \binom{d}{k+1} = \binom{d+1}{k+1}$.

Notice that this latter number in the displayed equation is the number of monomials of degree $m-1$ in $\mathbb{C}[x_0,\dots,x_n]$. □

Exercise 58.

The picture is sketched in the accompanying figure, and it is helpful to keep it in mind as you go through the solution. We will give two solutions – the first is very computational, and the second is very geometric (and possibly easier to follow).

First solution. By Exercise 41, the value of the Hilbert function is strictly increasing until it reaches the value 8, at which it stabilizes. It is clear that I_X contains no forms of degree 2.

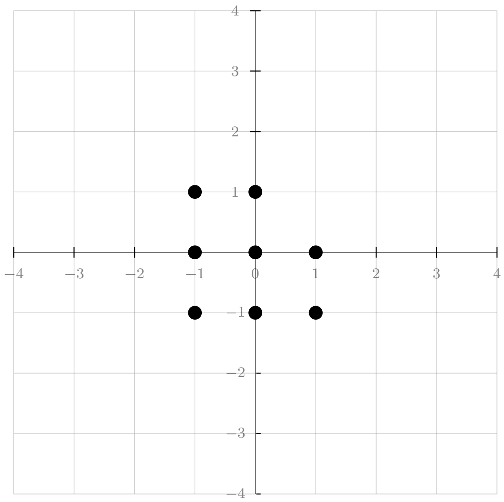


FIGURE 1. The set X in Exercises 58 and 59

Also note that the curves $(x+z)x(x-z)$ and $(y+z)y(y-z)) \in I_X$. So, there are only two possibilities either $H_X = (1,3,6,7,8,\ldots)$ or $H_X = (1,3,6,8,8,\ldots)$.

Let's start the computation.

$$I_X = \begin{array}{c} \langle x+z, y-z\rangle \cap \langle x, y-z\rangle \cap \\ \langle x+z, y\rangle \cap \langle x, y\rangle \cap \langle x-z, y\rangle \cap \\ \langle x+z, y+z\rangle \cap \langle x, y+z\rangle \cap \langle x-z, y+z\rangle. \end{array}$$

Then

$$\begin{aligned} I_X &= \Big\langle (x+z)x, y-z\Big\rangle \cap \Big\langle (x+z)x(x-z), y\Big\rangle \cap \Big\langle (x+z)x(x-z), y+z\Big\rangle \\ &= \Big\langle (x+z)x, y-z\Big\rangle \cap \Big\langle (x+z)x(x-z), y(y+z)\Big\rangle \\ &= \Big\langle (x+z)x(x-z), (y+z)y(y-z)), x(x+z)y(y+z)\Big\rangle. \end{aligned}$$

Now, the linear form z defines a line not containing any of the points in X so it is a regular element in R/I_X and then the first difference of H_X is the Hilbert function of the artinian algebra

$$R/\langle z\rangle/I_X + \langle z\rangle/\langle z\rangle = k[x,y]/\langle x^3, y^3, x^2y^2\rangle.$$

Therefore

$$\Delta H_X = (1,2,3,2)$$

and

$$H_X = (1,3,6,8,8,\ldots).$$

Second solution. Observe that X does not lie on any conics. We now show that X imposes independent conditions on $[R]_t$ for $t \geq 3$, and this will give the Hilbert function that we found in the first solution.

By Exercise 56, it is enough to show that the removal of any point $P_i \in X$ allows one to find a curve of degree $t \geq 3$ that vanishes at the remaining points but does not vanish at P_i. It is enough to handle the case $t = 3$. We leave it to you to check that for any such P_i there is a subset of the remaining points consisting of three collinear points, and then a conic vanishing at the remaining four points of $X \backslash \{P_i\}$ but not at P_i. $\square$

Exercise 59
In Exercise 58 we computed that $H_X(3) = 8$ and thus $\dim_k[I_X]_3 = 10 - 8 = 2$. However, the forms $(x+z)x(x-z), (y+z)y(y-z) \in I_X$ are generators for $[I_X]_3$ and both vanish at P. These forms define two cubic curves meeting in the 9 points of the set $X \cup \{P\}$, so P imposes no conditions on $[I_X]_3$; however, any other point not in this set will impose one condition on $[I_X]_3$. □

Exercise 60. Let $F \in [I_X]_t$, let P not in the hypersurface defined by F. Then P imposes a condition on forms of degree t vanishing at X since by construction, not every element of $[I_X]_t$ vanishes at P. □

Exercise 61. We have

$$I_X = (x,y)^2 \cap (x,z) \cap (y,z) = (xy, x^2z, y^2z).$$

So I_X is a monomial ideal and then the Hilbert function of X can be calculated as shown in Section 4. Therefore we get, $\dim[I_X]_4 = 10$. So e-dim$(X,4,4) = 10-10 = 0$. However, for a general point P of multiplicity 4, the curve C consisting of the union of the lines P_1P with multiplicity 2, P_2P and P_3P vanishes at X, and by Bezout's Theorem any curve of degree 4 vanishing at X and at $4P$ must be equal to C. Hence we get a-dim$(X,4,4) = 1$. □

Exercise 62. Assume that $a = 1$ or 2 and $b \geq 4$. The two lines $\ell_1, \ell_2 \in \mathcal{L}$ certainly lie on a smooth quadric surface (for instance since we know any three disjoint lines do), so the grid points X do as well. However, consider the grid lines $\ell'_1, \ell'_2, \ell'_3 \in \mathcal{L}'$. These determine a unique smooth quadric surface Q, and by Bezout's theorem this quadric surface must contain ℓ_1 and ℓ_2. But we have too much freedom to choose ℓ'_4 and beyond, and in particular they can be chosen off of Q.

On the other hand, if $3 \leq a \leq b$ then again considering $\ell'_1, \ell'_2, \ell'_3$ as before, we get that all of the other lines are forced to be on Q by Bezout's theorem. □

Exercise 63. Consider the plane spanned by three points in X and take a general point P on this plane. Project from P to get in $\mathbb{P}^2$ a set of six points, of which three are collinear and the other three are not on a line. Such a set of six points cannot lie on a conic. □

Exercise 64 Let π be the projection from a general point. If no three points of X are on a line and no four points are on a plane then, by Exercise 63, six of the points in X are enough to exclude that $\pi(X)$ lies on a conic. Therefore three points of X must be on a line. In order for X to be $(2,b)$-geproci, this line must be a component of the conic. Hence the conic containing $\pi(X)$ splits into the union of two lines, and again in order to be a complete intersection, both of the lines must contain b points of X. Since the projection is general and X is non-degenerate, X is contained in two skew lines. This is enough to conclude that X is a $(2,b)$-grid. □

14. Attached papers

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SURVEY ARTICLE: A TOUR OF THE WEAK AND STRONG LEFSCHETZ PROPERTIES

JUAN MIGLIORE AND UWE NAGEL

CONTENTS

ABSTRACT. An artinian graded algebra, A, is said to have the weak Lefschetz property (WLP) if multiplication by a general linear form has maximal rank in every degree. A vast quantity of work has been done studying and applying this property, touching on numerous and diverse areas of algebraic geometry, commutative algebra and combinatorics. Amazingly, though, much of this work has a "common ancestor" in a theorem originally due to Stanley, although subsequently reproved by others. In this paper we describe the different directions in which research has moved starting with this theorem, and we discuss some of the open questions that continue to motivate current research.

1. Introduction. The weak and strong Lefschetz properties are strongly connected to many topics in algebraic geometry, commutative

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algebra and combinatorics. Some of these connections are quite surprising and still not completely understood, and much work remains to be done. In this expository paper we give an overview of known results on the weak and strong Lefschetz properties, with an emphasis on the vast number of different approaches and tools that have been used, and connections that have been made with seemingly unrelated problems. One goal of this paper is to illustrate the variety of methods and connections that have been brought to bear on this problem for different families of algebras. We also discuss open problems.

Considering the amazing breadth and depth of results that have been found on this topic, and the tools and connections that have been associated with it, it is very interesting to note that, to a large degree, one result motivated this entire area. This result is the following. It was proved by Stanley [**53**] in 1980 using algebraic topology, by Watanabe in 1987 using representation theory, by Reid, Roberts and Roitman [**48**] in 1991 with algebraic methods, by Herzog and Popescu [**30**] (unpublished) in 2005, essentially with linear algebra, and it follows from the work of Ikeda [**50**] in 1996 using combinatorial methods.

Theorem 1.1. *Let $R = k[x_1, \dots, x_r]$, where k has* characteristic zero*. Let I be an artinian monomial complete intersection, i.e.,*

$$I = \langle x_1^{a_1}, \dots, x_r^{a_r} \rangle.$$

Let ℓ be a general linear form. Then, for any positive integers d and i, the homomorphism induced by multiplication by ℓ^d,

$$\times \ell^d : [R/I]_i \to [R/I]_{i+d}$$

has maximal rank. (*In particular, this is true when $d = 1$.*)

This paper is organized around the ways that subsequent research owes its roots to this theorem.

Our account is by no means exhaustive. Fortunately, the manuscript [**27**] has appeared recently. It gives an overview of the Lefschetz properties from a different perspective, focusing more on the local case, representation theory and combinatorial connections different from those presented here.

There is one topic that is neither treated in [**27**] nor here but that is worth mentioning briefly. In [**39**], examples of monomial ideals were exhibited that did not have the WLP but that could be deformed to ideals with the WLP. A systematic way for producing such deformations that preserve the Hilbert function has been proposed by Cook and Nagel in [**19**]. The idea is to lift the given monomial ideal to the homogenous ideal of a set of points and then pass to a general hyperplane section of the latter. It is shown in [**19**] that this procedure does indeed produce ideals with the WLP for a certain class of monomial ideals without the WLP.

In May 2011, the first author gave a talk at the Midwest Commutative Algebra and Geometry Conference at Purdue University on this topic. This paper is a vast expansion and extension of that talk, containing many more details and several new topics.

2. Definitions and background. Let k be an infinite field. We will often take $\operatorname{char}(k) = 0$, but we will see that changing the characteristic produces interesting new questions (and even more interesting answers!).

Let $R = k[x_1, \dots, x_r]$ be the graded polynomial ring in r variables over k. Let

$$A = R/I = \bigoplus_{i=0}^{n} A_i$$

be a graded artinian algebra. Note that A is finite dimensional over k.

Definition 2.1. For any standard graded algebra A (not necessarily artinian), the *Hilbert function* of A is the function

$$\underline{h}_A : \mathbf{N} \longrightarrow \mathbf{N}$$

defined by $\underline{h}_A(t) = \dim [A]_t$. One can express $\underline{h}_A$ as a sequence

$$(h_0 = 1, h_1, h_2, h_3, \dots).$$

An *O-sequence* is a sequence of positive integers that occurs as the Hilbert function of some graded algebra. When A is Cohen-Macaulay, its *h-vector* is the Hilbert function of an artinian reduction of A. In particular, when A is artinian, its Hilbert function is equal to its h-vector.

Definition 2.2. An *almost complete intersection* is a standard graded algebra $A = R/I$ which is Cohen-Macaulay, and for which the number of minimal generators of I is one more than its codimension.

Definition 2.3. A is *level of Cohen-Macaulay type* t if its socle is concentrated in one degree (e.g., a complete intersection) and has dimension t.

Definition 2.4. Let ℓ be a general linear form. We say that A has the *weak Lefschetz property* (WLP) if the homomorphism induced by multiplication by ℓ,

$$\times \ell : A_i \longrightarrow A_{i+1},$$

has maximal rank for all i (i.e., is injective or surjective). We say that A has the *strong Lefschetz property* (SLP) if

$$\times \ell^d : A_i \longrightarrow A_{i+d}$$

has maximal rank for all i and d (i.e., is injective or surjective).

Remark 2.5. (a) One motivation for the work described in this paper is that something interesting should be going on if multiplication by a general linear form does not induce a homomorphism of maximal rank, even in one degree.

(b) Later we will see that there is a strong connection to Fröberg's conjecture. In this regard, we note that ℓ^d should not be considered to be a "general" form of degree d, since in the vector space $[R]_d$ $(d > 1)$, those forms that are pure powers of linear forms form a proper Zariski-closed subset.

(c) Suppose that $\deg f = d$ and $\times f : [R/I]_i \rightarrow [R/I]_{i+d}$ has maximal rank, for all i. Pardue and Richert [**47**] call such an f *semi-regular*. Reid, Roberts and Roitman [**48**] call such an f *faithful*. If $\times f^j : [R/I]_i \rightarrow [R/I]_{i+dj}$ has maximal rank for all i and all j, they call such an f *strongly faithful*. So R/I has the WLP if R contains a linear faithful element, and R/I has the SLP if R contains a linear strongly faithful element.

(d) Several authors consider the question of the ranks that arise if $\times\ell^d$ is replaced by $\times F$ for a general F of degree d. This is the essence of the Fröberg conjecture, is related to the WLP, and will be discussed below in Section 6.

How do we determine if R/I fails to have the WLP? Let ℓ be a general linear form and fix an integer i. Then we have an exact sequence

$$[R/I]_{i-1} \stackrel{\times\ell}{\longrightarrow} [R/I]_i \longrightarrow [R/(I,\ell)]_i \longrightarrow 0.$$

Thus, $\times\ell$ fails to have maximal rank from degree $i-1$ to degree i if and only if

$$\dim [R/(I,\ell)]_i > \max\{0, \dim [R/I]_i - \dim [R/I]_{i-1}\}.$$

More precisely, if we want to show that the WLP fails, it is enough to identify a degree i for which we can produce one of the following two pieces of information:

(i) $\dim [R/I]_{i-1} \leq \dim [R/I]_i$ and $\dim [R/(I,\ell)]_i > \dim [R/I]_i - \dim [R/I]_{i-1}$; in this case, we loosely say that WLP *fails because of injectivity*; or

(ii) $\dim [R/I]_{i-1} \geq \dim [R/I]_i$ and $\dim [R/(I,\ell)]_i > 0$; in this case, we loosely say that WLP *fails because of surjectivity*.

In general, even identifying which i is the correct place to look can be difficult. Then, determining which of (i) or (ii) holds, and establishing both inequalities, is often very challenging. This is where computer algebra programs have been very useful, in suggesting where to look and what to look for! On the other hand, to prove that R/I *does* have the WLP, the following result is helpful:

Proposition 2.6 [**39**, Proposition 2.1]. *Let R/I be an artinian standard graded algebra, and let ℓ be a general linear form. Consider the homomorphisms $\phi_d : [R/I]_d \to [R/I]_{d+1}$ defined by multiplication by ℓ, for $d \geq 0$.*

(a) *If ϕ_{d_0} is surjective for some d_0, then ϕ_d is surjective for all $d \geq d_0$.*

(b) *If R/I is level and ϕ_{d_0} is injective for some $d_0 \geq 0$, then ϕ_d is injective for all $d \leq d_0$.*

(c) *In particular, if R/I is level and* $\dim [R/I]_{d_0} = \dim [R/I]_{d_0+1}$ *for some d_0 then R/I has the* WLP *if and only if ϕ_{d_0} is injective (and hence is an isomorphism).*

This result helps to narrow down where one has to look, especially in the situation where we want to show that the WLP *does* hold. In this case you have to find the critical degrees and *then* show that surjectivity *and* (usually) injectivity *do* hold just in two (or occasionally one) spots.

In the case of one variable, the WLP and SLP are trivial since all ideals are principal. The case of two variables also has a nice result, at least in characteristic 0:

Theorem 2.7 [**28**]. *If* $\operatorname{char}(k) = 0$ *and I is* any *homogeneous ideal in $k[x, y]$, then R/I has the* SLP*.*

The proof of this result used generic initial ideals with respect to the reverse lexicographic order. In the case of the WLP, it is not hard to show that the above theorem is true in any characteristic ([**20, 35, 44**]). However, the characteristic zero assumption cannot be omitted for guaranteeing the SLP. In fact, also the WLP may fail if there are at least three variables. The following is an easy exercise:

Lemma 2.8. *Assume* $\operatorname{char}(k) = p$*. Consider the ideal*

$$I = \langle x_1^p, \dots, x_r^p \rangle \subset R = k[x_1, \dots, x_r],$$

where $r \geq 2$. Then

- *$R/\langle x_1^p, \dots, x_r^p \rangle$ fails the* SLP *for all $r \geq 2$.*
- *It fails the* WLP *for all $r \geq 3$.*
- *It* has *the* WLP *when $r = 2$.*

Remark 2.9. It was pointed out to us by the referee that, in order for failure of SLP to hold, one does not even need to take the exponents to be p for all the variables. It suffices to take exponents at most

p summing to at least $p+r$. For example, $I = \langle x_1^{2+p-r}, x_2^2, \dots, x_r^2 \rangle$ works if $p \geq r$, and $I = \langle x_1^2, x_2^2, \dots, x_r^2 \rangle$ works if $p \leq r$. For the WLP one can as well use $I = \langle x_1^p, x_2^p, x_3^2, \dots, x_r^2 \rangle$.

In Section 7 we will discuss the presence of the WLP in positive characteristic in more detail.

A useful consequence of knowing that an algebra A has the WLP or SLP is that its Hilbert function is *unimodal*. In fact, the Hilbert functions of algebras with the WLP have been completely classified:

Proposition 2.10 [**28**]. *Let $\underline{h} = (1, h_1, h_2, \dots, h_s)$ be a finite sequence of positive integers. Then $\underline{h}$ is the Hilbert function of a graded artinian algebra with the* WLP *if and only if the positive part of the first difference is an O-sequence and after that the first difference is non-positive until $\underline{h}$ reaches* 0*. Furthermore, this is also a necessary and sufficient condition for $\underline{h}$ to be the Hilbert function of a graded artinian algebra with the* SLP*.*

The challenge is thus to study the WLP and SLP (or their failures), and the behavior of the Hilbert function, for interesting *families* of algebras. Most of the results below fall into this description. It should also be noted that, conversely, some Hilbert functions $\underline{h}$ *force* any algebra with Hilbert function $\underline{h}$ to have the WLP; these were classified in [**44**].

In the rest of this paper, we indicate different directions of research that have been motivated by Theorem 1.1; in most cases, there also remain many intriguing open problems.

3. Complete intersections and Gorenstein algebras. By semicontinuity, a consequence of Theorem 1.1 is that a *general* complete intersection with fixed generator degrees has the WLP and the SLP.

Question 3.1. *Do* all *artinian complete intersections have the* WLP *or the* SLP *in characteristic* 0?

We know that the answer is trivially "yes" in one and two variables.

In three or more variables, the following is the most complete result known to date.

Theorem 3.2 [**28**]. *Let* $R = k[x, y, z]$, *where* $\operatorname{char}(k) = 0$. *Let* $I = \langle F_1, F_2, F_3 \rangle$ *be a complete intersection. Then* R/I *has the* WLP.

The proof of this result introduced the use of the syzygy module of I, and its sheafification, the syzygy bundle. Subsequently, several papers have used the syzygy module to study the WLP for different kinds of ideals (see, e.g., [**11, 12, 20, 26, 39, 51**]). In the case of complete intersections in $k[x, y, z]$, the syzygy bundle has rank 2. The WLP is almost immediate in the "easy" cases, and semi-stability and the Grauert-Mülich theorem give the needed information about $R/(I, \ell)$ in the "interesting" cases.

Remark 3.3. (i) The SLP is still wide open for complete intersections in three or more variables and, in fact, even the WLP is open for complete intersections of arbitrary codimension ≥ 4. Some partial results on the WLP for arbitrary complete intersections in four variables have been obtained recently by the authors together with Boij and Miró-Roig, in work in progress.

(ii) It was conjectured by Reid, Roberts and Roitman [**48**] that the answer to both parts of Question 3.1 is yes.

We have seen that conjecturally (and known in special cases), all complete intersections have the WLP. Complete intersections are a special case of Gorenstein algebras. Does the conjecture extend to the Gorenstein case? That is,

Question 3.4. *Do all graded artinian Gorenstein algebras have the* WLP? *If not, what are classes of artinian Gorenstein algebras that do have this property*?

The answer to the first question is a resounding "no." Indeed, Stanley [**52**] in 1978 gave an example of an artinian Gorenstein algebra with Hilbert function $(1, 13, 12, 13, 1)$, which, because of non-unimodality, clearly does not have the WLP. Other examples of non-unimodality for Gorenstein algebras were given by Bernstein and Iarrobino [**3**], by Boij [**4**] and by Boij and Laksov [**5**]. Even among Gorenstein algebras with unimodal Hilbert functions, WLP does not necessarily hold. For instance, an example in codimension 4 was given by Ikeda [**32**] in 1996.

On the other hand, the problem in three variables is still wide open, with only special cases known (see for instance [**1, 45**]):

Question 3.5. *Does* every *artinian Gorenstein quotient of* $k[x,y,z]$ *have the* WLP*, provided* $\text{char}\,(k) = 0$*? What about the* SLP?

Given the complete intersection result for three variables mentioned above, this is a very natural and intriguing question.

In four variables, the result of Ikeda mentioned above shows that WLP need not hold. Nevertheless, the main result of [**43**] shows that, for small initial degree, the Hilbert functions are still precisely those of Gorenstein algebras with the WLP. More precisely, it was shown that, if the h-vector is $(1,4,h_2,h_3,h_4,\dots)$ and $h_4 \le 33$, then this result holds. More recently, using the same methods, Seo and Srinivasan [**51**] extended this to $h_4 = 34$. Thus, the result holds for initial degree ≤ 4.

Another interesting special case is the situation in which the generators of the ideal have small degree. We say that an algebra R/I is presented by quadrics if the ideal I is generated by quadrics. Such ideals occur naturally, for example, as homogeneous ideals of sufficiently positive embeddings of smooth projective varieties ([**23**]) or as Stanley-Reisner ideals of simplicial flag complexes ([**55**]). Gorenstein algebras presented by quadrics are studied, for example, in [**42**]. There, the following conjecture has been proposed.

Conjecture 3.6 [**42**]. *Any artinian Gorenstein algebra presented by quadrics, over a field* k *of characteristic zero, has the* WLP*.*

The conjecture predicts, in particular, that if the socle degree is at least 3 then the multiplication by a general linear form from degree one to degree two is injective. Though this is established in some cases in [**42**], even this special case of the conjecture is open.

The analog of Question 3.4 is also of interest for rings of positive dimension. If A is a Gorenstein ring of dimension d, then A is said to have the WLP if a general artinian reduction of A has the WLP, that is, if $A/\langle L_1, \dots, L_d\rangle$ has the WLP, where $L_1, \dots, L_d \in A$ are general forms of degree 1. Recall that the Stanley-Reisner ring of the boundary complex of a convex polytope is a reduced Gorenstein ring. The so-called g-theorem classifies their Hilbert functions. The necessity of the conditions on the Hilbert function is a consequence of the following result by Stanley.

Theorem 3.7 [**54**]. *The Stanley-Reisner ring of the boundary complex of a convex polytope over a field k has the* SLP *if* $\mathrm{char}\,(k) = 0$.

The so-called g-conjecture states that the above-mentioned conditions on the Hilbert function characterize in fact the Hilbert functions of the Stanley-Reisner rings of triangulations of spheres. Note that there are many more such triangulations than boundary complexes of convex polytopes. In this regard, the following question merits highlighting:

Question 3.8. *Does a general artinian reduction of a reduced, arithmetically Gorenstein set of points in $\mathbf{P}^n$ have the* WLP*, provided* $\mathrm{char}\,(k) = 0$?

We point out that, if this question has an affirmative answer, then, by the main result of [**41**], we have a classification of the Hilbert functions of reduced, arithmetically Gorenstein schemes: their h-vectors are precisely the SI-sequences, meaning that they are symmetric, with the first half itself a differentiable O-sequence.

An affirmative answer to Question 3.8 would also imply the g-conjecture, thus providing a characterization of the face vectors of triangulations of a sphere. Moreover, the methods used to establish the WLP could lead to information about the face vectors of triangulations of other manifolds as well. In fact, Novik and Swartz [**46**, Theorem 1.4], show that a certain quotient of the Stanley-Reisner ring of any orientable k-homology manifold without boundary is a Gorenstein ring. Kalai conjectured that this Gorenstein ring has the SLP. If true, this would establish new restrictions on the face vectors of these complexes.

A special case of Kalai's conjecture has been proved in Theorem 1.6 of [**46**].

4. Monomial level algebras. Note that $R/\langle x_1^{a_1}, \dots, x_r^{a_r}\rangle$ is also a level artinian monomial algebra.

Question 4.1. *Which (if any) level artinian monomial algebras fail the* WLP *or* SLP?

The first result in this direction is a positive one:

Theorem 4.2 (Hausel [**29**, Theorem 6.2]). *Let A be a monomial artinian level algebra of socle degree e. If the field k has characteristic zero, then for a* general *linear form ℓ, the induced multiplication*

$$\times \ell : A_j \longrightarrow A_{j+1}$$

is an injection, for all $j = 0, 1, \dots, \lfloor (e-1)/2 \rfloor$. In particular, over any field the sequence

$$1, h_1 - 1, h_2 - h_1, \dots, h_{\lfloor (e-1)/2\rfloor + 1} - h_{\lfloor (e-1)/2 \rfloor}$$

is an O-sequence, i.e., the "first half" of $\underline{h}$ is a differentiable *O-sequence.*

Thus, roughly "half" the algebra does satisfy the WLP. What about the second half? The first counterexample was due to Zanello ([**57**, Example 7]), who showed that the WLP does not necessarily hold for monomial level algebras even in three variables. His example had h-vector $(1, 3, 5, 5)$. Subsequently, Brenner and Kaid ([**11**, Example 3.1]) produced an example of a level artinian monomial *almost complete intersection* algebra that fails the WLP; this algebra has h-vector $(1, 3, 6, 6, 3)$ and, in particular, Cohen-Macaulay type 3. The study of such almost complete intersections was continued by Migliore, Miró-Roig and Nagel [**39**], and more recently by Cook and Nagel [**18, 20**] (see also Section 7).

The Hilbert functions of the algebras considered in Question 4.1 are of great interest in a number of areas. In fact, they are better known under a different name.

Definition 4.3. A *pure O-sequence of type t in r variables* is the Hilbert function of a level artinian monomial algebra $k[x_1, \dots, x_r]/I$ of Cohen-Macaulay type t.

Question 4.4. *We have already seen that level artinian monomial algebras do not necessarily have the* WLP. *Nevertheless, are their Hilbert functions unimodal? That is, are all pure O-sequences unimodal? If not, can we find subfamilies, depending upon the type t and/or the number of variables r, that are unimodal? And, if they are not necessarily unimodal, "how non-unimodal" can they be?*

Remark 4.5. If I is a *monomial ideal* in $R = k[x_1, \dots, x_r]$, then the linear form $\ell = x_1 + \cdots + x_r$ is "general enough" to determine whether R/I has the WLP or SLP. This observation has been extremely useful in simplifying calculations to show the existence or failure of the WLP. In [**39**, Proposition 2.2], this was stated for the WLP, but the same proof also gives it for the SLP.

For the remainder of this section we will *assume that k has characteristic* 0, unless explicitly mentioned otherwise. We have seen that, in one or two variables, we always have the WLP (and even SLP). Turning to the next case, the following seemingly simple result in fact has a very intricate and long proof. It illustrates the subtlety of these problems.

Theorem 4.6 [**8**, Theorem 6.2]. *A level artinian monomial algebra of type* 2 *in three variables has the* WLP.[1]

Of course, this has the following consequence.

Corollary 4.7. *A pure O-sequence of type* 2 *in three variables is unimodal.*

The monograph [**6**] gave a careful study of families of level artinian monomial algebras that fail the WLP. As a consequence, we have the following conclusion.

Theorem 4.8 [**6**]. *If* $R = k[x_1, \dots, x_r]$ *and* R/I *is a level artinian monomial algebra of type* t*, then, for all* r *and* t*, examples exist where the WLP fails, except if*:

- $r = 1$ *or* 2;
- $t = 1$ (*this is Theorem* 1.1);
- $r = 3$, $t = 2$ (*this is Theorem* 4.6).

In particular, the first case where WLP can fail is when $r = 3$ and $t = 3$. This occurs, for instance, if $I = \langle x^3, y^3, z^3, xyz \rangle$ (see [**11**, Example 3.1]). Nevertheless, Boyle has shown that, despite the failure of the WLP, all level artinian monomial algebras with $r = 3$ and $t = 3$ have *strictly unimodal* Hilbert function (that is, in addition to being unimodal, once the function decreases then it is strictly decreasing from that point until it reaches zero):

Theorem 4.9 [**9**]. *Any pure O-sequence of* type 3 *in three variables is strictly unimodal.*

In more variables, the first case where the WLP can fail is when $r = 4$ and $t = 2$. Here again, Boyle has shown that, nevertheless, such algebras have strictly unimodal Hilbert function:

Theorem 4.10 [**10**]. *Any pure O-sequence of type* 2 *in four variables is strictly unimodal.*

Since the WLP is not available in these cases, Boyle's method is a classification theorem followed by a decomposition of the ideals and a careful analysis of sums of Hilbert functions of complete intersections.

However, there is no hope of such a result for all pure O-sequences, even when $r = 3$:

Theorem 4.11 [**8**]. *Let* M *be any positive integer, and fix an integer* $r \geq 3$*. Then there exists a pure O-sequence in* r *variables which is non-unimodal, having exactly* M *maxima.*

In view of the last two results, we have the following natural question.

Question 4.12. *What is the smallest socle degree and* (*separately*) *the smallest socle type t for which non-unimodal pure O-sequences exist? This is especially of interest when $r = 3$.*

In [**6**], Boij and Zanello produced a non-unimodal example with $r = 3$ and socle degree 12. In [**8**], for $r = 3$, we produced a non-unimodal example for socle type $t = 14$. It was also shown that pure O-sequences can fail unimodality if and only if the socle degree is at least 4 (but one may need many variables for small socle degree).

It is also natural to ask how things change when you remove "monomial" and ask about artinian level algebras. Some work in progress by Boij, Migliore, Miró-Roig, Nagel and Zanello indicates that the behavior of such algebras from the point of view of the Hilbert function can become surprisingly worse, in the sense that dramatic non-unimodality is possible even in early degrees, which would violate Hausel's theorem (Theorem 4.2) for instance, in the monomial case.

5. Powers of linear forms. In this section we always assume that k has characteristic zero. Note that x_i is a linear form, and that if $L_1, \dots, L_n$ ($n \geq r$) are general linear forms, then, without loss of generality, (by a change of variables) we can assume that $L_1 = x_1, \dots, L_r = x_r$. Thus, Theorem 1.1 is also a result about ideals generated by powers of linear forms. It says that, in $k[x_1, \dots, x_r]$, an ideal generated by powers of r general linear forms has the WLP and the SLP. It also leads to an interesting connection to Fröberg's conjecture, which we discuss in Section 6.

Question 5.1. *Which ideals generated by powers of general linear forms define algebras that fail the* WLP *or* SLP?

We saw in Theorem 2.7 that all such ideals (and in fact all homogeneous ideals) in two variables satisfy both the WLP and the SLP. More surprisingly, Schenck and Seceleanu showed a similar result in three variables:

Theorem 5.2 [**51**]. *Let* $R = k[x, y, z]$, *where* $\mathrm{char}\,(k) = 0$. *Let* $I = \langle L_1^{a_1}, \dots , L_m^{a_m}\rangle$ *be* any *ideal generated by powers of linear forms. Then* R/I *has the* WLP.

A shorter proof of this result is given in [**40**]. One reason that it is surprising is that the same is *not* true for SLP. For instance, if $I = \langle L_1^3, L_2^3, L_3^3, L_4^3\rangle$ (where L_i is general in $k[x, y, z]$), then $(\times \ell^3)$ fails to have maximal rank. The case of three variables acts as a bridge case: we will see that, for four or more variables, even WLP fails very commonly. Some recent work in this area was motivated by the following example of Migliore, Miró-Roig and Nagel:

Example 5.3 [**39**]. Let $r = 4$. Consider the ideal $I = \langle x_1^N, x_2^N, x_3^N, x_4^N, L^N\rangle$ for a general linear form L. By computation using CoCoA, R/I fails the WLP, for $N = 3, \dots , 12$.

There are some natural questions arising from this example:

Problem 5.4. • *Prove the failure of the* WLP *in Example* 5.3 *for all* $N \geq 3$.

• *What happens for mixed powers*?

• *What happens for almost complete intersections, that is, for* $r + 1$ *powers of general linear forms in* r *variables when* $r > 4$?

• *What about more than* $r + 1$ *powers of general linear forms*?

This example motivated two different projects at the same time: by Migliore, Miró-Roig, Nagel [**40**] and by Harbourne, Schenck and Seceleanu [**26**]. Both of these papers used the dictionary between ideals of powers of general linear forms and ideals of fat points in projective space, provided by the following important result of Emsalem and Iarrobino:

Theorem 5.5 [**24**]. *Let*

$$\langle L_1^{a_1}, \dots , L_n^{a_n}\rangle \subset k[x_1, \dots , x_r]$$

be an ideal generated by powers of n *linear forms. Let* $\wp_1, \dots , \wp_n$ *be* *the ideals of the n points in $\mathbf{P}^{r-1}$ corresponding to the linear forms. Then, for any integer $j \geq \max\{a_i\}$,*

$$\dim_k [R/\langle L_1^{a_1}, \dots, L_n^{a_n} \rangle]_j = \dim_k \left[\wp_1^{j-a_1+1} \cap \cdots \cap \wp_n^{j-a_n+1} \right]_j .$$

One important difference between the two papers is that [**26**] assumed that the powers are uniform, and usually that the powers are "large enough." Usually they allow more than $r+1$ forms. On the other hand, most of the results in [**40**] allow mixed powers. We quote some of the results of these two papers. Together they form a nice start to an interesting topic. The conjectures listed later indicate that more work is to be done!

Theorem 5.6 [**26**]. *Let*

$$I = \langle L_1^t, \dots, L_n^t \rangle \subset k[x_1, x_2, x_3, x_4],$$

with $L_i \in R_1$ generic. If $n \in \{5, 6, 7, 8\}$, then the WLP *fails, respectively, for $t \geq \{3, 27, 140, 704\}$.*

Theorem 5.7 [**26**]. *For*

$$I = \langle L_1^t, \dots, L_{2k+1}^t \rangle \subset R = k[x_1, \dots, x_{2k}]$$

with L_i generic linear forms, $k \geq 2$ and $t \gg 0$, R/I fails the WLP*.*

(See also Theorem 5.10 below.) The following result gives the most complete picture to date, about the case of four variables, when the exponents are not assumed to be uniform and the ideal is assumed to be an almost complete intersection (i.e. the number of minimal generators is one more than the number of variables). It summarizes several theorems in [**40**, Section 3], and we refer to that paper for the more detailed individual statements.

Theorem 5.8 (Four variables, [**40**]). *Let*

$$I = \langle L_1^{a_1}, L_2^{a_2}, L_3^{a_3}, L_4^{a_4}, L_5^{a_5} \rangle \subset R = k[x_1, x_2, x_3, x_4],$$

where all L_i are generic. Without loss of generality, assume that $a_1 \le a_2 \le a_3 \le a_4 \le a_5$. Set

$$\lambda = \begin{cases} (a_1+a_2+a_3+a_4)/2-2 & \text{if } a_1+a_2+a_3+a_4 \text{ is even} \\ (a_1+a_2+a_3+a_4-7)/2 & \text{if } a_1+a_2+a_3+a_4 \text{ is odd.} \end{cases}$$

(a) *If $a_5 \ge \lambda$, then R/I has the* WLP.

(b) *If $a_1 = 2$, then R/I has the* WLP.

(c) *Most other cases (explicitly described in terms of a_1, a_2, a_3, a_4) are proven to fail the* WLP.

(d) *For the few open cases, experimentally sometimes the* WLP *holds and sometimes it does not.*

Notice that the case where all the a_i are equal and at least 3 is contained in Theorem 5.6.

In more than four variables, it becomes progressively more difficult to obtain results for mixed powers. We have the following partial result.

Theorem 5.9 (Five variables, almost uniform powers [**40**]). *Assume $r = 5$. Let $L_1, \dots, L_6$ be general linear forms. Let $e \ge 0$ and*

$$I = \langle L_1^d, L_2^d, L_3^d, L_4^d, L_5^d, L_6^{d+e} \rangle.$$

(a) *If $e = 0$, then R/I fails the* WLP *if and only if $d > 3$.*

(b) *If $e \ge 1$ and d is odd, then R/I has the* WLP *if and only if $e \ge (3d-5)/2$.*

(c) *If $e \ge 1$ and d is even, then R/I has the* WLP *if and only if $e \ge (3d-8)/2$.*

We also have the following improvement of Theorem 5.7, which has the additional assumption that $t \gg 0$.

Theorem 5.10 (Even number of variables, uniform powers [**40**]). *Let*

$$I = \langle L_1^t, \dots, L_{2k+1}^t \rangle \subset R = k[x_1, \dots, x_{2k}]$$

with L_i generic linear forms and $k \geq 2$. Then R/I fails the WLP *if and only if $t > 1$.*

(The case $k = 2$ is contained in Theorem 5.8.)

What about an odd number of variables? Here is a result for seven variables:

Theorem 5.11 [**40**]. *Let*

$$I = \langle L_1^t, \ldots, L_8^t \rangle \subset k[x_1, \ldots, x_7],$$

where $L_1, \ldots, L_8$ are general linear forms.

- *If $t = 2$, then R/I has the* WLP*.*
- *If $t \geq 4$, then R/I fails the* WLP*.*

Interestingly, for $t = 3$, CoCoA [**16**] says that the WLP fails, but we do not have a proof. We can believe a computer that says that the WLP *holds*, but otherwise we have to be skeptical about whether its choice of forms was "general enough."

For these results, sometimes it was necessary to prove *failure of surjectivity* (when $h_{i-1} \geq h_i$ in the relevant degrees), sometimes *failure of injectivity* (when $h_{i-1} \leq h_i$), and sometimes we had to show that *the* WLP *does hold*. These present quite different challenges.

After making the translation to fat points, as described above, the first difficulty is to determine the degrees where WLP fails. Then, it is necessary to find the dimension of a linear system of surfaces in a suitable projective space vanishing to prescribed multiplicity at a general set of suitably many points. To do this, in [**40**], Cremona transformations and works of Dumnicki (2009), Laface-Ugaglia (2006) and De Volder-Laface (2007) were used as the main tools, plus ad hoc methods. These Cremona transformation results are central to the proofs in [**40**].

Much remains to be shown on this topic. Here are two conjectures from [**26, 40**].

Conjecture 5.12 [**26**]. *For* $I = \langle L_1^t, \dots, L_n^t \rangle \subset R = [x_1, \dots, x_r]$ *with* $L_i \in R_1$ *generic and* $n \geq r+1 \geq 5$*, the* WLP *fails for all* $t \gg 0$.

Conjecture 5.13 [**40**]. *Let* $R = k[x_1, \dots, x_{2n+1}]$. *Let* $L_1, \dots, L_{2n+2}$ *be general linear forms and* $I = \langle L_1^d, \dots, L_{2n+1}^d, L_{2n+2}^d \rangle$.

- *If* $n = 3$ *and* $d = 3$*, then* R/I *fails the* WLP. (*This is the only open case in Theorem* 5.11.)
- *If* $n \geq 4$*, then* R/I *fails the* WLP *if and only if* $d > 1$.

These conjectures are supported by a great deal of computer evidence, using CoCoA [**16**] and `Macaulay2` [**25**].

6. Connection between Fröberg's conjecture and the WLP. In this section we continue to assume that our field has characteristic zero. Closely related to the SLP is the so-called *maximal rank property* (MRP), which just replaces ℓ^d by a general form of degree d in Definition 2.4. Nevertheless, it is known that the MRP does not imply the SLP. See [**38, 58**] for some connections between these two properties.

One way of stating Fröberg's conjecture is as follows.

Conjecture 6.1 (Fröberg). *Any ideal of general forms has the* MRP. *More precisely, fix positive integers* $a_1, \dots, a_s$ *for some* $s > 1$. *Let* $F_1, \dots, F_s \subset R = k[x_1, \dots, x_r]$ *be general forms of degrees* $a_1, \dots, a_s$*, respectively, and let* $I = \langle F_1, \dots, F_s \rangle$. *Then, for each* i*,* $2 \leq i \leq s$*, and for all* t*, the multiplication by* F_i *on* $R/\langle F_1, \dots, F_{i-1} \rangle$ *has maximal rank, from degree* $t - a_i$ *to degree* t. *As a result, the Hilbert function of* R/I *can be computed inductively.*

This conjecture is known to be true in two variables. This follows, for example, from Theorem 2.7. In three variables it was shown to be true by Anick [**2**]. In this section, we explore the following natural questions.

Question 6.2. *What is the Hilbert function of an ideal generated by powers of general linear forms of degrees* $a_1, \dots, a_n$? *In particular, is* *it the same as the Hilbert function predicted by Fröberg? What, if any, is the connection to the WLP?*

Theorem 1.1 says that, when $n = r + 1$, the answer to the second question is yes.

The fact that the answer is often "no" for $n = r+2$ was first observed by Iarrobino [**31**]. Chandler [**13, 14**] also gave some results in this direction. Concerning the connection to the WLP, the following result of Migliore, Miró-Roig and Nagel gives a partial answer.

Proposition 6.3 [**40**]. (a) *If Fröberg's conjecture is true for all ideals generated by general forms in r variables, then all ideals generated by general forms in $r+1$ variables have the* WLP.

(b) *Let $R = k[x_1, \dots, x_{r+1}]$, let $\ell \in R$ be a general linear form and let $S = R/\langle \ell \rangle \cong k[x_1, \dots, x_r]$. Fix positive integers $s, d_1, \dots, d_s, d_{s+1}$. Let $L_1, L_2, \dots, L_s, L_{s+1} \in R$ be linear forms. Denote by $\overline{}$ the restriction from R to $S \cong R/\langle \ell \rangle$. Make the following assumptions*:

(i) *The ideal $I = \langle L_1^{d_1}, \dots, L_s^{d_s} \rangle$ has the* WLP.

(ii) *The multiplication $\times \overline{L}_{s+1}^{d_{s+1}} : [S/\overline{I}]_{j-d_{s+1}} \to [S/\overline{I}]_j$ has maximal rank.*

Then $R/\langle L_1^{d_1}, \dots, L_{s+1}^{d_{s+1}} \rangle$ has the WLP.

Remark 6.4. (a) Part of this result was in fact contained in the paper [**38**] of Migliore and Miró-Roig. It was used there to show that any ideal of general forms in $k[x_1, x_2, x_3, x_4]$ satisfies the WLP, because Anick [**2**] had shown much earlier that any ideal of general forms in $k[x_1, x_2, x_3]$ satisfies Fröberg's conjecture.

(b) It was shown in [**40**] that this result also leads to a short proof of Theorem 5.2. The point is that the restriction of such ideals corresponds to an ideal in $k[x, y]$, and in characteristic zero all such ideals have the SLP by Theorem 2.7.

The following corollary was also shown in [**40**]:

Corollary 6.5 [**40**]. *Assume the characteristic is zero. Let $R = k[x_1, \dots, x_{r+1}]$, let $\ell \in R$ be a general linear form and let $S = R/\langle \ell \rangle \cong k[x_1, \dots, x_r]$. For integers $d_1, \dots, d_{r+2}$, if an ideal of the* *form* $\langle L_1^{d_1}, \dots, L_{r+2}^{d_{r+2}} \rangle \subset R$ *of powers of general linear forms fails to have the* WLP*, then an ideal of powers of general linear forms* $\langle \overline{L}_1^{d_1}, \dots, \overline{L}_{r+2}^{d_{r+2}} \rangle \subset S$ *fails to have the Hilbert function predicted by Fröberg's conjecture.*

Thus, the results in the previous section give additional insight to the observations of Iarrobino [**31**] and Chandler [**13, 14**] that, when $n = r + 2$, there are many cases when an ideal of powers of general linear forms does not have the same Hilbert function as that predicted by Fröberg for general forms. Since Theorem 5.8 covers almost all possible choices of exponents, it gives a much more complete answer to the question of exactly which powers of five general linear forms in three variables fail to have the Fröberg-predicted Hilbert function, contrasting with the result of Anick which says that an ideal of general forms of any fixed degrees in three variables does have the predicted Hilbert function. Theorems 5.9 and 5.10 provide new partial answers (via Corollary 6.5) in the case of more variables.

Example 6.6. Let $R = k[x_1, x_2, x_3, x_4]$. Let L_1, L_2, L_3, L_4, L_5 and ℓ be general linear forms. Let $S = R/\langle L \rangle \cong k[x, y, z]$. Let $I = \langle L_1^3, L_2^3, L_3^3, L_4^3, L_5^3 \rangle$ (the smallest case in Example 5.3 above). The Hilbert function of R/I is $(1, 4, 10, 15, 15, 6)$. We have

$$[R/I]_3 \xrightarrow{\times \ell} [R/I]_4 \longrightarrow [R/(I, \ell)]_4 \longrightarrow 0.$$

We saw that WLP fails, and in fact

$$\dim\, [R/(I, \ell)]_4 = 1.$$

Notice that $R/(I, \ell) \cong S/J$, where J is the ideal of cubes of five general linear forms in $k[x, y, z]$. Thus, $\dim\, [S/J]_4 = 1$.

On the other hand, let K be the ideal of five general cubics in S. Fröberg predicts (and Anick proves) that $\dim\, [S/K]_4 = 0$. Thus, J does not have the Hilbert function predicted by Fröberg.

In fact, whenever we prove that an ideal of n powers of general linear forms fails the WLP (for specified exponents), then for some subset of these powers of general linear forms, the same number and powers of general linear forms in one fewer variable fails to have Fröberg's predicted Hilbert function.

7. Positive characteristics and enumerations. Considering Theorem 1.1 again, we saw in Lemma 2.8 that the assumption on the characteristic of the base field cannot be omitted.

Question 7.1. *What happens in Theorem* 1.1 *if we allow the characteristic to be positive*?

Actually, investigating the dependence of the WLP on the characteristic makes sense whenever the ideal can be defined over the integers. This applies to all monomial ideals. In fact, in this case one has the following result.

Proposition 7.2 [**20**, Lemma 2.6]. *Let* $I \subset R$ *be a monomial ideal. If* R/I *has the* WLP *when* $\operatorname{char}(k) = 0$*, then* R/I *has the* WLP *whenever* $\operatorname{char}(k)$ *is sufficiently large.*

The proof is based on two observations that have their origin in [**39**]. For a monomial ideal, one can check the WLP by considering the specific linear form $\ell = x_1 + \cdots + x_r$. Thus, the maximal rank property of the multiplications by ℓ is governed by integer matrices. Their determinants have only finitely many prime divisors if they do not vanish.

It also follows that R/I fails the WLP in *every* positive characteristic if it fails the WLP in characteristic zero.

Proposition 7.2 motivates the following problem.

Question 7.3. *Let* $I \subset R$ *be a monomial ideal such that* R/I *has the WLP when* $\operatorname{char}(k) = 0$. *What are the* (*finitely many*) *field characteristics such that* R/I *fails the* WLP?

This turns out to be a rather subtle problem. It was first considered in [**39**] in the case of a certain almost complete intersection in three variables. Recall that a monomial almost complete intersection in three variables is an ideal of the form

$$I = I_{a,b,c,\alpha,\beta,\gamma} = \langle x^a, y^b, z^c, x^\alpha y^\beta z^\gamma \rangle. \tag{7.1}$$

If the syzygy bundle of I is not semi-stable or its first Chern class is not divisible by three, then R/I has the WLP in characteristic zero (see [**11, 39**]). However, if the syzygy bundle satisfies both conditions, then deciding the WLP is more difficult and very subtle on the one side. On the other side, the investigations in this case have brought to light surprising connections to combinatorial problems.

In fact, if the syzygy bundle of I is semi-stable and its first Chern class is divisible by three, then R/I has the WLP if and only if the multiplication by ℓ in a certain degree is an isomorphism or, equivalently, a certain integer square matrix has a non-vanishing determinant. This has first been observed in the special case, where R/I is level, in [**39**] and then for arbitrary almost complete intersections in [**20**]. The first connection to combinatorics was made by Cook and Nagel in [**18**, Section 4]. There it was observed that the determinant deciding the WLP for certain families of monomial almost complete intersections is the number of lozenge tilings of some hexagon, which is given by a formula of MacMahon. Lozenge tilings of a hexagon are in bijection to other well-studied combinatorial objects such as, for example, plane partitions and families of non-intersecting lattice paths.

Independently of [**18**], but subsequent to it, Li and Zanello studied the WLP in the case of the complete intersections $R/\langle x^a, y^b, z^c \rangle$ in [**35**], and they also related MacMahon's numbers of plane partitions to the failure of the WLP:

Theorem 7.4 [**35**]. *For any given positive integers a, b, c, the number of plane partitions contained inside an $a \times b \times c$ box is divisible by a prime p if and only if the algebra $k[x,y,z]/\langle x^{a+b}, y^{a+c}, z^{b+c} \rangle$ fails to have the* WLP *when* $\operatorname{char}(k) = p$.

(This connection is already implicitly contained in [**18**], although it was only made explicit in the proof of Corollary 6.5 in [**20**].) Next, Chen, Guo, Jin and Liu [**15**], explained *bijectively* the result by Li and Zanello for complete intersections. Both [**18, 35**] have been substantially extended in [**20**]. Here the bijective approach of [**15**] was extended to almost complete intersections, and further relations between the presence of the WLP and difficult counting problems in combinatorics have been given. In the remainder of this section, we give an overview of some of the results of [**20**] which illustrate this fascinating connection.

We focus on the most difficult case, in which the presence of the WLP is a priori not even known in characteristic zero, that is, we assume that the syzygy bundle of the almost complete intersection $I = I_{a,b,c,\alpha,\beta,\gamma} = \langle x^a, y^b, z^c, x^\alpha y^\beta z^\gamma \rangle$ is semi-stable in characteristic zero and its first Chern class is divisible by three. By [**20**, Proposition 3.3], this is exactly true if and only if the following conditions are all satisfied:

(i) $s := (a+b+c+\alpha+\beta+\gamma)/3 - 2$ is an integer,

(ii) $0 \le M$,

(iii) $0 \le A \le \beta + \gamma$,

(iv) $0 \le B \le \alpha + \gamma$, and

(v) $0 \le C \le \alpha + \beta$,

where

$$\begin{aligned} A &:= s+2-a, \\ B &:= s+2-b, \\ C &:= s+2-c, \text{ and} \\ M &:= s+2-(\alpha+\beta+\gamma). \end{aligned}$$

The above conditions have a geometric meaning. In fact, due to Theorem 4.1 in [**20**], they guarantee that I can be related to a hexagonal region with a hole, which is called the *punctured hexagon* $H = H_{a,b,c,\alpha,\beta,\gamma}$ associated to $I = I_{a,b,c,\alpha,\beta,\gamma}$ (see Figure 1).

There are two square matrices that govern the WLP of the ideal I. In fact, I has the WLP if and only if the multiplication $[R/I]_s \stackrel{\times \ell}{\to} [R/I]_{s+1}$ is bijective or, equivalently, $[R/(I,\ell)]_{s+1} = 0$. The latter condition means that a certain $(C+M) \times (C+M)$ matrix, $N = N_{a,b,c,\alpha,\beta,\gamma}$, with binomial coefficients as entries is regular. The above multiplication map can be described by a much larger zero-one square matrix, $Z = Z_{a,b,c,\alpha,\beta,\gamma}$. The above-mentioned equivalence implies that the determinants of N and Z have the same prime divisors. However,

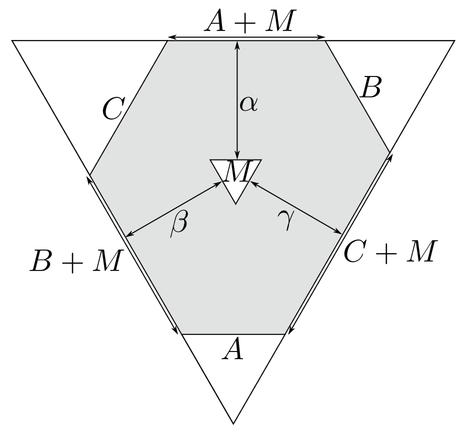


FIGURE 1. Punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}$ (shadowed) associated to $I_{a,b,c,\alpha,\beta,\gamma}$.

much more is true. Both determinants have the same absolute value, which has combinatorial interpretations.

Theorem 7.5 [**20**, Theorems 4.4, 4.6 and 5.4]. *Adopt the above assumptions. Then the following conditions are equivalent*:

(a) $I_{a,b,c,\alpha,\beta,\gamma}$ *has the* WLP *if the characteristic of the base field* k *is* $p \geq 0$.

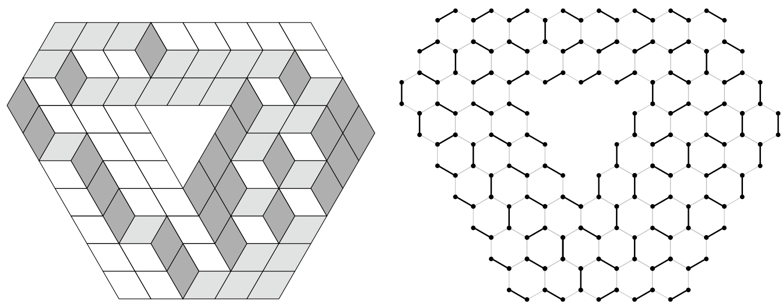
(b) p *does not divide the enumeration* $|\det N_{a,b,c,\alpha,\beta,\gamma}|$ *of signed lozenge tilings of the associated punctured hexagon* $H_{a,b,c,\alpha,\beta,\gamma}$.

(c) p *does not divide the enumeration* $|det\, Z_{a,b,c,\alpha,\beta,\gamma}|$ *of signed perfect matchings of the bipartite graph associated to* $H_{a,b,c,\alpha,\beta,\gamma}$.

In particular, $|\det N_{a,b,c,\alpha,\beta,\gamma}| = |\det Z_{a,b,c,\alpha,\beta,\gamma}|$.

A lozenge is a rhombus with unit side-lengths and angles of 60° and 120°. Lozenges have also been called calissons and diamonds in the literature. A perfect matching of a graph is a set of pairwise non-adjacent edges such that each vertex of the graph is matched. We refer to [**20**] for more details, in particular for assigning the signs, although Figure 2 indicates an associated lozenge tiling and a perfect matching.

Theorem 7.5 has been used to establish the WLP of $I_{a,b,c,\alpha,\beta,\gamma}$ in many new cases. The results also lend further evidence to a conjectured characterization of the presence of the WLP of $I_{a,b,c,\alpha,\beta,\gamma}$ in case $I_{a,b,c,\alpha,\beta,\gamma}$ is level that has been proposed in [**39**].



Hexagon tiling by lozenges Perfect matching of edges

FIGURE 2. A lozenge tiling and its associated perfect matching.

The determinants occurring in Theorem 7.5 can be rather big.

Example 7.6. Consider the ideal

$$I = \langle x^{14}, y^{21}, z^{25}, x^2y^9z^{13} \rangle.$$

Then the absolute value of the corresponding determinants is (see [**18**, Remark 4.8])

$$2 \cdot 3^2 \cdot 5^3 \cdot 11^4 \cdot 13^5 \cdot 19 \cdot 23^3 \cdot 29 \cdot 5011.$$

Hence, R/I fails the WLP if and only if the characteristic of k is any of the nine listed prime divisors.

In the situation of Theorem 7.5, the presence of the WLP in characteristic zero can also be read off from the splitting type of the syzygy bundle. In fact, $I_{a,b,c,\alpha,\beta,\gamma}$ has the WLP if and only if its syzygy bundle has splitting type $(s+2, s+2, s+2)$ (see [**20**, Theorem 9.9]).

In [**20**], explicit formulae for the enumerations appearing in Theorem 7.5 are derived in various cases. However, even then determining the prime divisors of the enumerations can be challenging. In fact, this problem is open even in the special case of monomial complete intersections, though, recently, there has been progress in the case where the generators all have the same degree. Brenner and Kaid [**12**] gave an explicit description of when $R/\langle x^d, y^d, z^d \rangle$ has the WLP in terms of

d and the characteristic p. In particular, they proved a conjecture of [**35**] for the case $p = 2$. This latter result is stated very concisely:

Theorem 7.7 [**12**]. *The algebra* $k[x, y, z]/\langle x^d, y^d, z^d\rangle$ *has the* WLP *in* char $(k) = 2$ *if and only if* $d = \lfloor (2^n + 1)/3 \rfloor$ *for some positive integer* n.

The approach of [**12**] was via a theorem of Han computing the syzygy gap for an ideal of the form $\langle x^d, y^d, (x + y)^d\rangle$ in $k[x, y]$. The analogous result in the case of more variables, that is, for $I = \langle x_1^d, \ldots, x_n^d\rangle$ $(n \geq 4)$, has been obtained by Kustin and Vraciu in [**34**]. Independently, Cook made progress in deciding the Lefschetz properties of more general monomial complete intersections in positive characteristic (see [**17**]), addressing Question 7.1 (see also [**36**, Lemma 5.2], for a result in two variables).

In a different direction, Kustin, Rahmati and Vraciu [**33**] showed that $A = R/\langle x^d, y^d, z^d\rangle$ has the WLP in characteristic $p \neq 2$ if and only if its residue field has finite projective dimension as an A-module.

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ENDNOTES

1. A simpler proof of this result has recently been given in [**21**].

Note added in proof. In the time since this paper was submitted, several important advances have been made in the study of the WLP which we have not been able to describe in this survey. Among these we single out [**7, 21, 22, 37**].

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The Weak and Strong Lefschetz properties for Artinian K-algebras

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Abstract

Let $A = \bigoplus_{i \geqslant 0} A_i$ be a standard graded Artinian K-algebra, where char $K = 0$. Then A has the Weak Lefschetz property if there is an element ℓ of degree 1 such that the multiplication $\times\ell : A_i \to A_{i+1}$ has maximal rank, for every i, and A has the Strong Lefschetz property if $\times\ell^d : A_i \to A_{i+d}$ has maximal rank for every i and d. The main results obtained in this paper are the following.

(1) *Every* height-three complete intersection has the Weak Lefschetz property. (Our method, surprisingly, uses rank-two vector bundles on $\mathbb{P}^2$ and the Grauert–Mülich theorem.)

(2) We give a complete characterization (including a concrete construction) of the Hilbert functions that can occur for K-algebras with the Weak or Strong Lefschetz property (and the characterization is the same one!).

(3) We give a sharp bound on the graded Betti numbers (achieved by our construction) of Artinian K-algebras with the Weak or Strong Lefschetz property and fixed Hilbert function. This bound is again the same for both properties! Some Hilbert functions in fact *force* the algebra to have the maximal Betti numbers. (4) *Every* Artinian ideal in $K[x, y]$ possesses the Strong Lefschetz property. This is false in higher codimension.

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1. Introduction

Let A be a graded Artinian algebra over some field K (which we will restrict shortly). Then A has the *Weak Lefschetz property* (sometimes called the Weak Stanley property) if there is an element ℓ of degree 1 such that the multiplication $\times\ell : A_i \to A_{i+1}$ has maximal rank, for every i. We say that A has the *Strong Lefschetz property* if there is an element ℓ of degree 1 such that the multiplication $\times\ell^d : A_i \to A_{i+d}$ has maximal rank for every i and d. If $A = R/I$, where R is a polynomial ring and I is a homogeneous ideal, then sometimes we will abuse notation and refer to the Weak or Strong Lefschetz properties for I rather than for A. These are both fundamental properties and have been studied by many authors, especially when A is Gorenstein (e.g., [4,13,15,16,19,24,26–28]).

Throughout this paper, unless specified otherwise, we assume that we work over a field of characteristic zero. This paper began with a study of the Weak Lefschetz property for complete intersections of height three, and grew to a study of Artinian ideals of arbitrary codimension. Our original interest in the subject was to try to get a handle on "how many" Artinian complete intersections possess this natural property. However, a further motivation comes from the fact that this property can be translated into (at least) two other natural questions.

First, suppose that $F_1, F_2, \ldots, F_n$ is a homogeneous complete intersection in the n-dimensional polynomial ring R. Then the minimal free resolution of the ideal $(F_1, \ldots, F_n)$ is well understood; namely it is obtained as the Koszul complex. However, the graded Betti numbers of the minimal free resolution of the ideal $(F_1, \ldots, F_n, L)$, where L is a generic linear form, does not seem to be well understood. For example, should they be all the same, depending only on the degrees of the generators and not on the generators themselves, as long as they are a regular sequence of given degrees plus a generic element? (We could also ask the same question for L^d in the place for L.) The connection between the Weak Lefschetz property and this question is discussed in the last part of Section 2, and we give a complete answer (Corollary 2.7) when $n = 3$.

One other problem concerns the generic initial ideal, $\mathrm{gin}(I)$, of a complete intersection I, i.e., the initial ideal of I with respect to generic variables (cf., for instance, [9]). It is well known that $\mathrm{gin}(I)$ is Borel-fixed. But if I is a complete intersection and if we fix a monomial order, is the Borel-fixed ideal $\mathrm{gin}(I)$ unique? Or are there two complete intersections I and J such that $\mathrm{gin}(I)$ and $\mathrm{gin}(J)$ are different Borel-fixed ideals with the same Hilbert function? These questions seem to be open since if $\mathrm{gin}(I)$ is unique with respect to the reverse lex order then it would imply the Strong Lefschetz property of all complete intersections of those degrees. Since a Borel-fixed ideal is unique in codimension two (for a fixed Hilbert function), the Strong Lefschetz property can be proved in this case (Proposition 4.4).

It should also be mentioned that Stanley and others have made deep connections between the Weak and Strong Lefschetz properties and questions in combinatorics [24, 25]. For example, the Weak Lefschetz property was the crucial ingredient in Stanley's part of the characterization of the f-vectors of simplicial polytopes. Thus, we are exploring in this paper also the restrictions on the possible Hilbert functions and graded Betti numbers imposed by the presence of the Weak or Strong Lefschetz property.

It was noticed by Stanley [25] and independently by the fourth author [27] that any monomial complete intersection (in any number of variables) has the Strong Lefschetz property, and the fourth author proved that in any codimension, "most" Artinian Gorenstein rings with fixed socle degree possess the Strong Lefschetz property [27, Example 3.9]. We remark (following [15]) that Stanley's proof used the idea of recognizing $A = R/I$ as the cohomology ring of a product X of projective spaces, and then using the hard Lefschetz theorem for the algebraic variety X. The fourth author noticed that it follows from the representation theory of the Lie algebra $sl(2)$.

Yet even in codimension 3, we do not have a clear idea of which Artinian Gorenstein rings possess this property, and in particular whether all of them do. The (apparently) simplest situation is for height-3 complete intersections in $R = K[x_1, x_2, x_3]$. Until now the most general result for this case is again due to the fourth author. Suppose that the generators of the complete intersection I have degrees $2 \leqslant d_1 \leqslant d_2 \leqslant d_3$. Then it was proved in [28] that if $d_3 > d_1 + d_2 - 2$ then R/I has the Weak Lefschetz property. But for arbitrary complete intersections, even the case of three polynomials of degree 4 had been open.

The first main result of this paper (Theorem 2.3) is that *all* Artinian complete intersections in $K[x_1, x_2, x_3]$ have the Weak Lefschetz property. It is a somewhat surprising result. Indeed, it was known to be a very difficult problem among the experts, and at times it seemed more natural to seek a counter-example rather than to try to prove it! We are able to give a relatively simple proof by translating the problem to one of vector bundles on $\mathbb{P}^2$ and invoking a deep theorem due to Grauert and Mülich.

This part of the paper was inspired by [28], but as mentioned earlier, our techniques are completely different from those of the papers cited above. Because we apply the Grauert–Mülich theorem, we are forced to assume characteristic zero (as indeed was done in [28]). In fact, the Weak Lefschetz property does not hold for all complete intersections in characteristic p; see Remark 2.9.

As a further illustration of the power of our approach, we give a simple proof (Corollary 2.5) of the main result of [28].

In the third section of the paper we do not assume that $\operatorname{char} K = 0$. We consider graded Artinian K-algebras which are not necessarily complete intersections. Here we produce (Construction 3.4) a particular graded Artinian K-algebra, which allows us to give a necessary and sufficient condition for a sequence of integers to be the Hilbert function of a graded Artinian K-algebra with the Weak Lefschetz property (Proposition 3.5). We also answer several natural questions about the minimal free resolutions of algebras with the Weak Lefschetz property.

Our second main result (Theorem 3.20) shows that if we fix an allowable Hilbert function then there is a sharp upper bound on the graded Betti numbers among K-algebras having the Weak Lefschetz property. Indeed, this bound is achieved by the algebra produced by Construction 3.4, once we refine the construction slightly. This result is analogous to the main result of [19], which proved it for Gorenstein ideals with the Weak Lefschetz property (see also [11]). As a corollary we show that there are Hilbert functions which occur for K-algebras with the Weak Lefschetz property and for which this property forces the graded Betti numbers to be the maximal ones.

In Section 4 we again assume char $K = 0$. We consider the Strong Lefschetz property, namely that there exists a linear form ℓ such that for each d, the multiplication $\times \ell^d : A_i \to A_{i+d}$ has maximal rank, for every i. This condition implies the Weak Lefschetz property, but is not equivalent to it in general. We show that these conditions are both automatic in codimension two, however.

Since there are algebras with the Weak Lefschetz property but not the Strong Lefschetz property, one might guess that the imposition of the Strong Lefschetz property reduces the number of possible Hilbert functions. However, we are able to show that with another slight refinement of Construction 3.4, that algebra has the Strong Lefschetz property. This yields the surprising result that a Hilbert function occurs among algebras with the Weak Lefschetz property if and only if it occurs among algebras with the Strong Lefschetz property. Furthermore, the extremal graded Betti numbers for algebras with the Weak Lefschetz property also occur among algebras with the Strong Lefschetz property.

Our results have some consequences for the punctual Hilbert scheme. Since by semicontinuity the Weak and Strong Lefschetz properties are open properties, it follows that the general point of a component has the Strong (respectively Weak) Lefschetz property if and only if the component has one point with the Strong (respectively Weak) Lefschetz property. Moreover, we know precisely the possible Hilbert functions of the K-algebras corresponding to such a general point.

2. The Weak Lefschetz Property for height-three complete intersections

Let $R = K[x_1, x_2, x_3]$, where K is a field of characteristic zero. Initially we will assume that K is algebraically closed, in order to freely use the results of [23]. However, we note in Corollary 2.4 and beyond that our results hold without that assumption.

Let I be a complete intersection ideal of R generated by homogeneous elements $F_1, F_2, F_3 \in R$ of degrees d_1, d_2, d_3 respectively, and $d_1 \leqslant d_2 \leqslant d_3$. The minimal free resolution for R/I has the form

$$\begin{array}{ccccccccccccc} 0 & \to & R(-d_1-d_2-d_3) & \to & \mathbb{F}_2 & & \longrightarrow & & \mathbb{F}_1 & \xrightarrow{[F_1, F_2, F_3]} & R & \to & R/I \to 0 \\ & & & & & \searrow & & \nearrow & & & & & \\ & & & & & & E & & & & & & \\ & & & & & \nearrow & & \searrow & & & & & \\ & & & & 0 & & & & 0 & & & & \end{array} \tag{2.1}$$

where $\mathbb{F}_2 = R(-d_2-d_3) \oplus R(-d_1-d_3) \oplus R(-d_1-d_2)$ and $\mathbb{F}_1 = R(-d_1) \oplus R(-d_2) \oplus R(-d_3)$. Sheafifying, we get the following two exact sequences:

$$0 \to \mathcal{E} \to \mathcal{F}_1 \xrightarrow{[F_1, F_2, F_3]} \mathcal{O}_{\mathbb{P}^2} \to 0 \tag{2.2}$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d_1-d_2-d_3) \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{E} \longrightarrow 0, \tag{2.3}$$

where $\mathcal{E}$ is locally free (since I is Artinian) of rank two, $\mathcal{F}_1 = \mathcal{O}_{\mathbb{P}^2}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_2) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_3)$ and $\mathcal{F}_2 = \mathcal{O}_{\mathbb{P}^2}(-d_2 - d_3) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_1 - d_3) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2)$.

We would like a condition which forces $\mathcal{E}$ to be semistable. We first consider the case where $d_1 + d_2 + d_3$ is even. Choose an integer d so that $2d = d_1 + d_2 + d_3$. Notice that $c_1(\mathcal{E}) = -d_1 - d_2 - d_3 = -2d$, so the normalized bundle $\mathcal{E}_{\text{norm}}$ is $\mathcal{E}(d)$ (an easy computation, or see, for instance, [23, p. 165]). Twisting the sequence (2.3) by $d - 1$ we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d-1) \longrightarrow \begin{matrix} \mathcal{O}_{\mathbb{P}^2}(-d+d_1-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(-d+d_2-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(-d+d_3-1) \end{matrix} \longrightarrow \mathcal{E}_{\text{norm}}(-1) \longrightarrow 0. \tag{2.4}$$

We now consider the case where $d_1 + d_2 + d_3$ is odd. Choose d so that $2d = d_1 + d_2 + d_3 - 1$. Then again $\mathcal{E}_{\text{norm}} = \mathcal{E}(d)$ (again see [23, p. 165]). Now we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d-1) \longrightarrow \begin{matrix} \mathcal{O}_{\mathbb{P}^2}(-d+d_1-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(-d+d_2-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(-d+d_3-1) \end{matrix} \longrightarrow \mathcal{E}_{\text{norm}} \longrightarrow 0. \tag{2.5}$$

Lemma 2.1. *Let $\mathcal{E}$ be the rank-two locally free sheaf obtained above as the kernel of the map $[F_1, F_2, F_3]$.*

(1) *Assume $d_1 + d_2 + d_3$ is even. If $d_3 < d_1 + d_2 + 2$ then $\mathcal{E}$ is semistable.*
(2) *Assume $d_1 + d_2 + d_3$ is odd. If $d_3 < d_1 + d_2 + 1$ then $\mathcal{E}$ is semistable.*

Proof. When $c_1(\mathcal{E})$ is even and $\mathcal{E}$ has rank two, we know from [23, Lemma 1.2.5] that $\mathcal{E}$ is semistable if and only if $H^0(\mathbb{P}^n, \mathcal{E}_{\text{norm}}(-1)) = 0$ (since it has rank two). When $c_1(\mathcal{E})$ is odd and $\mathcal{E}$ has rank two, stability and semistability coincide [23, p. 166] and the condition for semistability is $H^0(\mathbb{P}^2, \mathcal{E}_{\text{norm}}) = 0$.

The two sequences (2.4) and (2.5) are exact on global sections. Hence semistability follows in either case if we have $-d + d_3 - 1 < 0$ (where d changes slightly depending on the parity of $d_1 + d_2 + d_3$). The lemma then follows from a simple computation. □

Let $\lambda \cong \mathbb{P}^1$ be a general line in $\mathbb{P}^2$. Recall that every vector bundle on $\mathbb{P}^1$ splits, so in particular $\mathcal{E}|_\lambda \cong \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2)$. The pair (e_1, e_2) is called the *splitting type* of $\mathcal{E}$.

Corollary 2.2. *Let $\mathcal{E}$ be the locally free sheaf obtained above, and assume that $d_3 < d_1 + d_2 + 1$. Then the splitting type of $\mathcal{E}$ is*

$$(e_1, e_2) = \begin{cases} (-d, -d) & \text{if } d_1 + d_2 + d_3 = 2d; \\ (-d, -d-1) & \text{if } d_1 + d_2 + d_3 - 1 = 2d. \end{cases}$$

Proof. By Lemma 2.1, $\mathcal{E}$ is semistable. The theorem of Grauert and Mülich ([23, p. 206], [8, p. 68]) says that in characteristic zero the splitting type of a semistable normalized 2-bundle $\mathcal{E}_{\text{norm}}$ over $\mathbb{P}^n$ is

$$(e_1, e_2) = \begin{cases} (0,0) & \text{if } c_1(\mathcal{E}_{\text{norm}}) = 0; \\ (0,-1) & \text{if } c_1(\mathcal{E}_{\text{norm}}) = -1. \end{cases}$$

In our case $\mathcal{E}_{\text{norm}} = \mathcal{E}(d)$, so a simple calculation gives the result. □

With this preparation, we now prove the main result of the paper. We continue to assume that K is algebraically closed of characteristic zero.

Theorem 2.3. *Every height-three Artinian complete intersection has the Weak Lefschetz property.*

Proof. It was shown in [28, Corollary 3] that if $d_3 \geqslant d_1 + d_2 - 3$ then R/I has the Weak Lefschetz property. So without loss of generality assume that $d_3 < d_1 + d_2 - 3$. Note that then Corollary 2.2 applies. To prove the Weak Lefschetz property it is enough to prove injectivity in the "first half," so we will focus on this.

Let L be a general linear form and let $\overline{R} = R/L$. We denote by $\overline{F}$ the restriction of a polynomial F to $\overline{R}$ and by $\overline{\mathbb{F}}_1$ the free $\overline{R}$-module $\overline{R}(-d_1) \oplus \overline{R}(-d_2) \oplus \overline{R}(-d_3)$. Consider the multiplication induced by L. From (2.1) we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & E(-1) & \to & \mathbb{F}_1(-1) & \xrightarrow{[F_1\ F_2\ F_3]} & R(-1) & \to & R/I(-1) & \to & 0 \\
 & & & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle (\times L)} & & \downarrow{\scriptstyle (\times L)} & & \\
0 & \longrightarrow & E & \longrightarrow & \mathbb{F}_1 & \xrightarrow{[F_1\ F_2\ F_3]} & R & \longrightarrow & R/I & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \overline{\mathbb{F}}_1 & \xrightarrow{[\overline{F}_1\ \overline{F}_2\ \overline{F}_3]} & \overline{R} & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array} \tag{2.6}$$

where M is the matrix

$$\begin{bmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & L \end{bmatrix}.$$

Note that the first vertical exact sequence is the direct sum of three copies of the exact sequence

$$0 \longrightarrow R(-1) \xrightarrow{\times L} R \longrightarrow \overline{R} \longrightarrow 0$$

twisted by $-d_1$, $-d_2$, and $-d_3$, respectively. The induced map on the kernels, $E(-1) \to E$, is just multiplication by L.

Let λ be the line in $\mathbb{P}^2$ defined by L. Invoking the Snake Lemma and using the fact that the sheafification of R/I is 0, the sheafified version of (2.6) is

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}(-1) & \longrightarrow & \mathcal{F}_1(-1) & \xrightarrow{[F_1\ F_2\ F_3]} & \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\times L)} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle (\times L)} & & \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}_1 & \xrightarrow{[F_1\ F_2\ F_3]} & \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}|_\lambda & \longrightarrow & \overline{\mathcal{F}}_1 & \xrightarrow{[\overline{F}_1\ \overline{F}_2\ \overline{F}_3]} & \mathcal{O}_\lambda & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.7}
$$

By Corollary 2.2,

$$
\mathcal{E}|_\lambda \cong \begin{cases} \mathcal{O}_\lambda(-d)^2, & \text{if } d_1 + d_2 + d_3 = 2d; \\ \mathcal{O}_\lambda(-d) \oplus \mathcal{O}_\lambda(-d-1), & \text{if } d_1 + d_2 + d_3 - 1 = 2d. \end{cases}
$$

Let $\bar{I}$ be the ideal $(\overline{F}_1, \overline{F}_2, \overline{F}_3)$ in $\overline{R}$. Taking global sections on the last line of (2.7) gives

$$
0 \longrightarrow \bigoplus_{i=1}^{2} \overline{R}(-e_i) \longrightarrow \overline{\mathbb{F}}_1 \longrightarrow \bar{I} \longrightarrow 0
$$

where $|e_1 - e_2| = 0$ or 1 according to whether $d_1 + d_2 + d_3$ is even or odd, respectively. It was observed in [28, Remark on p. 3165] that this implies that R/I has the Weak Lefschetz property. However, for completeness we will sketch the argument. We will treat only the case $d_1 + d_2 + d_3$ even, leaving the other case to the reader.

We have the exact sequence

$$
0 \longrightarrow \overline{R}(-d)^2 \longrightarrow \begin{array}{c} \overline{R}(-d_1) \\ \oplus \\ \overline{R}(-d_2) \\ \oplus \\ \overline{R}(-d_3) \end{array} \xrightarrow{[\overline{F}_1\ \overline{F}_2\ \overline{F}_3]} \bar{I} \longrightarrow 0
\tag{2.8}
$$

where $d = (d_1 + d_2 + d_3)/2$.

As noted earlier, we only have to show that multiplication by L is injective on the "first half" of R/I. The socle degree of R/I is $d_1 + d_2 + d_3 - 3$, so we have to show that the multiplication

$$(R/I)_j \xrightarrow{\times L} (R/I)_{j+1}$$

is injective for $j \leqslant (d_1 + d_2 + d_3)/2 - 2 = d - 2$. We will show it to be true for $j = d - 2$, and from the form of the proof it will be clear that it holds also for smaller j.

The kernel of $(\times L)$ is $[I :_R L]$, so if $(\times L)$ is not injective we have an element $F \in R_{d-2}$, $F \notin I$, such that $LF \in I_{d-1}$. That is, we have forms A_i, $1 \leqslant i \leqslant 3$, with $\deg A_i = d - 1 - d_i$ and

$$LF - A_1F_1 - A_2F_2 - A_3F_3 = 0.$$

Restricting this syzygy to $\overline{R}$ gives

$$\overline{A}_1\overline{F}_1 + \overline{A}_2\overline{F}_2 + \overline{A}_3\overline{F}_3 = 0.$$

But (2.8) says that the smallest possible syzygies come from polynomials of degree $d - d_i$, $1 \leqslant i \leqslant 3$, so this is a contradiction. As noted, this works equally well to prove injectivity for all $j \leqslant d - 2$. $\square$

Corollary 2.4. *Let K be a field of characteristic zero which is not necessarily algebraically closed. Then every height-three Artinian complete intersection in $K[x_1, x_2, x_3]$ has the Weak Lefschetz property.*

Proof. The Weak Lefschetz property for a graded Artinian K-algebra A is equivalent to the statement that for a general linear form ℓ, the Hilbert function of $A/\ell A$ is just the positive part of the first difference of the Hilbert function of A. But this does not change under extension of the base field, so the result follows from Theorem 2.3.

Using the same methods, we can also give a new proof of the main result of [28]. As above, we can assume that K is algebraically closed initially, but the rest of the results of this section do not need this assumption.

Corollary 2.5. *Let $R = K[x_1, x_2, x_3]$, $I = (F_1, F_2, F_3)$ a complete intersection in R, $d_i = \deg F_i$ for $i = 1, 2, 3$, L a general linear form, $\overline{R} = R/LR$, and $\bar{I} = (I + LR)/LR$. Then the following are equivalent*:

(i) $\mu(\bar{I}) = 3$, *where μ is the minimal number of generators*;
(ii) $d_3 \leqslant d_1 + d_2 - 2$.

Proof. For completeness we repeat the proof from [28] of the fact that (i) implies (ii). Since L is general, F_1, F_2, and L are regular sequences, and the socle degree of $R/(F_1, F_2, L)$

is $d_1 + d_2 - 2$. If (ii) is not true then F_3 is contained in the ideal (F_1, F_2, L), so $\overline{F}_3$ is contained in $((F_1, F_2) + LR)/LR$, contradicting (i).

The hard part of the proof is the converse, which we prove using our approach. We have from Corollary 2.2 that the splitting type of $\mathcal{E}$ is

$$(e_1, e_2) = \begin{cases} (-d, -d) & \text{if } d_1 + d_2 + d_3 = 2d; \\ (-d, -d-1) & \text{if } d_1 + d_2 + d_3 - 1 = 2d. \end{cases}$$

With this definition of d, a simple calculation gives that

$$\text{If } d \text{ is even then} \quad d_3 < d \quad \Leftrightarrow \quad d_3 < d_1 + d_2;$$

$$\text{If } d \text{ is odd then} \quad d_3 < d \quad \Leftrightarrow \quad d_3 < d_1 + d_2 - 1.$$

So in either case, if (ii) holds then $d_3 < d$. But the splitting type gives exactly the leftmost free module in the short exact sequence (2.8), and the fact that $d_3 < d$ means that no splitting can occur in the resolution. □

We now apply these ideas to the question of minimal free resolutions. In particular, suppose $I = (F_1, F_2, F_3)$ is a complete intersection in $R = K[x_1, x_2, x_3]$ and F is a general form of degree d. What can be the possible minimal free resolutions of the ideal (I, F)? Does it depend only on the degrees of the generators of I, or does the choice of the complete intersection itself play a role? We can answer this question when F has degree 1, which in any case was an open question. To be consistent with notation, we write L for this general linear polynomial. We begin with a lemma.

Lemma 2.6. *Let $I \subset R = K[x_1, x_2, x_3]$ be an Artinian ideal. Then there exists a Cohen–Macaulay height-two ideal $J \subset R$ such that $J + (L) = I + (L)$. J can even be taken to be reduced.*

Proof. Let $I = (F_1, \dots, F_k)$. We know that $I + (L)/(L) = (\overline{F}_1, \dots, \overline{F}_k)$ is Artinian in $\overline{R} = R/(L)$, hence Cohen–Macaulay of height 2. After a change of coordinates, we can assume that $L = x_3$, hence we obtain polynomials $G_1, \dots, G_k \in K[x_1, x_2]$ by canceling all monomials in $F_1, \dots, F_k$ which are a multiple of x_3. Then viewing these polynomials in R gives the first result. This ideal is not reduced, however. But it has a Hilbert–Burch matrix, whose entries are all polynomials in x_1, x_2. Using standard lifting techniques, one can obtain a reduced scheme J with the desired property. (A more geometric use of this trick may be found in [6].) □

Note that the preceding lemma trivially implies that all the graded Betti numbers (over $R/(L)$) of the reduction of J modulo L are the same as those of the reduction of I modulo L. However, in general we are not able to say what these Betti numbers are, or what the Betti numbers of the ideal $I + (L)$ are (over R), or even what the Hilbert function is. Nevertheless, in the case of complete intersections we can say something much stronger, thanks to our results above.

Corollary 2.7. *Let $I = (F_1, F_2, F_3) \subset R$ be a complete intersection. Then there is a (reduced) arithmetically Cohen–Macaulay ideal $J = (G_1, G_2, G_3) \subset R$ such that $\deg G_i = \deg F_i = d_i$ for $i = 1, 2, 3$ and such that $J + (L) = I + (L)$. Furthermore,*

(a) *If $d_3 \leqslant d_1 + d_2 - 2$ then J is an almost complete intersection with minimal generators given by the G_i. Let d be defined by*

$$\begin{cases} d_1 + d_2 + d_3 = 2d & \text{if } d_1 + d_2 + d_3 \text{ is even;} \\ d_1 + d_2 + d_3 - 1 = 2d & \text{if } d_1 + d_2 + d_3 \text{ is odd.} \end{cases}$$

If $d_1 + d_2 + d_3$ is even then the minimal free resolution of $R/(I + (L))$ is given by

$$0 \longrightarrow R(-d-1)^2 \longrightarrow \begin{matrix} R(-d_1-1) \\ \oplus \\ R(-d_2-1) \\ \oplus \\ R(-d_3-1) \\ \oplus \\ R(-d)^2 \end{matrix} \longrightarrow \begin{matrix} R(-1) \\ \oplus \\ R(-d_1) \\ \oplus \\ R(-d_2) \\ \oplus \\ R(-d_3) \end{matrix} \longrightarrow R \longrightarrow R/(I + (L)) \longrightarrow 0.$$

(*The case where $d_1 + d_2 + d_3$ is odd is analogous.*)

(b) *If $d_3 > d_1 + d_2 - 2$ then $J = (G_1, G_2)$ is a complete intersection. In this case the minimal free resolution of $R/(I + (L))$ is given by*

$$0 \to R(-d_1-d_2-1) \longrightarrow \begin{matrix} R(-d_1-1) \\ \oplus \\ R(-d_2-1) \\ \oplus \\ R(-d_1-d_2) \end{matrix} \longrightarrow \begin{matrix} R(-1) \\ \oplus \\ R(-d_1) \\ \oplus \\ R(-d_2) \end{matrix} \longrightarrow R \longrightarrow R/(I + (L)) \to 0.$$

Proof. The first part of the corollary is immediate from Lemma 2.6.

For both (a) and (b) we know that $(I + (L)) = (J + (L))$ where J is arithmetically Cohen–Macaulay of depth 1. Hence $R/(I + (L))$ has the same resolution as $R/(J + (L))$, either over R or over $R/(L)$.

Consider (a). We know from Corollary 2.5 that $[I + (L)]/(L) = [J + (L)]/(L) \subset R/(L)$ is an almost complete intersection, so the same is true of $J \subset R$ since $\operatorname{depth} R/J = 1$. Suppose that $d_1 + d_2 + d_3$ is even (the case where it is odd is completely analogous). We have a minimal free resolution (over $R/(L)$) for $R/(J + (L))$ given in Theorem 2.3, so we thus have a minimal free resolution over R for R/J given by

$$0 \longrightarrow R(-d)^2 \longrightarrow \begin{matrix} R(-d_1) \\ \oplus \\ R(-d_2) \\ \oplus \\ R(-d_3) \end{matrix} \longrightarrow R \longrightarrow R/J \longrightarrow 0.$$

Then the desired minimal free resolution for $R/(J+(L))$ (and hence $R/(I+(L)))$ is given by the tensor product of this resolution with the resolution

$$0 \longrightarrow R(-1) \longrightarrow R \longrightarrow R/(L) \longrightarrow 0.$$

The proof of (b) is trivial □

Remark 2.8. It is possible that similar techniques can be used to prove the Strong Lefschetz property for height-three complete intersections (see Definition 4.1), or to attack either the Weak or Strong Lefschetz properties for Artinian complete intersections in higher-dimensional rings. However, a more subtle proof will be needed, as simple examples show that the *degrees* of the syzygies will not be enough to obtain a contradiction.

Nevertheless, *we conjecture that every Artinian complete intersection in* $K[x_0, x_1, x_2]$ *has the Strong Lefschetz property*.

Remark 2.9. What happens in characteristic p? We first note that we cannot expect a result as strong as the one given in Theorem 2.3. Indeed, let $A = K[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2)$ where K has characteristic 2. Let $g = ax_1 + bx_2 + cx_3$ be a general linear form. Then $g : A_1 \to A_2$ is not injective; indeed, g is itself in the kernel! A similar observation can be made for $A = K[x_1, x_2, x_3]/(x_1^4, x_2^4, x_3^4)$, etc.

The main problem here is that the Grauert–Mülich theorem does not hold in characteristic p. There are weaker versions: a theorem of Ein [8, Theorem 4.1] bounds the splitting type of $\mathcal{E}$ by a function of $c_2(\mathcal{E})$. However, as we saw in the proof of Theorem 2.3, we need the full strength of Grauert–Mülich in order to prove our result. In the highest degree (at the "middle" of the h-vector), the contradiction from the degrees of the syzygies would not have occurred if this degree had been one greater. Hence a weaker version of Grauert–Mülich is not good enough with the present techniques.

For example, if I is the complete intersection of three polynomials of degree 10 in R, then one can compute that $c_2(\mathcal{E}_{\text{norm}}) = 75$, and then Ein's theorem gives that the splitting type is no worse than $(5, -5)$. However, that means that the restriction to $\overline{R} = R/(L)$ has resolution "no worse" than

$$0 \longrightarrow \overline{R}(-10) \oplus \overline{R}(-20) \longrightarrow \overline{R}(-10)^3 \longrightarrow \bar{I} \longrightarrow 0.$$

In particular, it cannot even be excluded that the restriction of I to $\overline{R}$ is a complete intersection. In characteristic zero this is excluded immediately by our work above (applying the strong Grauert–Mülich theorem) and in fact it follows immediately also from the main theorem of [28].

Remark 2.10. (1) The Weak Lefschetz property says that a general linear form induces a map of maximal rank on consecutive components. One might be interested in a description of the set of (special) linear forms which does *not* give maps of maximal rank. This is parameterized by the variety of jumping lines of the bundle $\mathcal{E}$.

It is interesting to combine the two techniques involved here. For any set of distinct lines $\lambda_1, \ldots, \lambda_r$ in $\mathbb{P}^2$, one can easily construct bundles having the λ_i as jumping lines. For example, let $r = 3$ and consider complete intersections of type $(4, 4, 4)$.

On λ_i, $i = 1, 2, 3$, choose general points $P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,1}, Q_{i,2}, R_{i,1}, R_{i,2}, R_{i,3}, R_{i,4}$. Consider the 4-tuples

$$(P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,1}), \quad (P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,2}), \quad \text{and} \quad (R_{i,1}, R_{i,2}, R_{i,3}, R_{i,4}).$$

Choose a general quartic curve $F_1 \in R_4$ containing the 12 points $(P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,1})$ $(i = 1, 2, 3)$, a general quartic curve $F_2 \in R_4$ containing the 12 points $(P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,2})$ $(i = 1, 2, 3)$, and a general quartic curve $F_3 \in R_4$ containing the 12 points $(R_{i,1}, R_{i,2}, R_{i,3}, R_{i,4})$ $(i = 1, 2, 3)$. (This is possible since the points were chosen generically.)

Then (F_1, F_2, F_3) is a complete intersection, but its restrictions to λ_1, λ_2, and λ_3 each have linear syzygies. Let $\mathcal{E}$ be the bundle constructed from this complete intersection. Since the restriction to a general line has no smaller than quadratic syzygies, λ_1, λ_2, and λ_3 are jumping lines.

(2) The bundle $\mathcal{E}$ used in this section is a Buchsbaum–Rim sheaf. The interested reader can find a much more extensive treatment of such sheaves and their properties in [17,18, 20].

3. Hilbert functions and maximal Betti numbers of algebras with the Weak Lefschetz property

In this section we do not require that $\operatorname{char} K = 0$ or that K be algebraically closed. We give a complete characterization of the possible Hilbert functions of algebras with the Weak Lefschetz property. Furthermore, we show that there is a sharp upper bound on all of the graded Betti numbers in the minimal free resolution of an algebra with the Weak Lefschetz property. For the remainder of this paper we write $R = K[x_0, \ldots, x_n]$.

Notation 3.1. If $A = R/I$ is a graded K-algebra then we denote the Hilbert function of A by

$$h_A(t) := \dim_K [R/I]_t.$$

The main result of [13] was to characterize the Gorenstein sequences (i.e., the sequences of integers that can arise as the Hilbert function of an Artinian Gorenstein ideal) corresponding to Artinian Gorenstein ideals with the Weak Lefschetz property. These turned out to be the so-called Stanley–Iarrobino (SI)-sequences. As a consequence, since the height-three Gorenstein ideals are well understood ([5,7] among others), in $K[x_1, x_2, x_3]$ every Gorenstein sequence occurs as the Hilbert function of an Artinian ideal with the Weak Lefschetz property. We now consider the non-Gorenstein case.

Question 3.2. Which Hilbert functions (in any codimension) can occur for ideals whose coordinate rings have the Weak Lefschetz property?

We will give a complete answer to this question, giving a construction for an Artinian K-algebra with any allowable Hilbert function, having the Weak Lefschetz property. Later we will give a bound for the graded Betti numbers of an Artinian K-algebra with the Weak Lefschetz property (Theorem 3.20), and we will show that our construction achieves the bound. Of course, this result includes as a special case the maximal possible socle type. However, we have the additional nice result that this maximal socle type can be read directly from the Hilbert function, so we will consider the socle type along with the Hilbert function.

Let A be an Artinian graded K-algebra with the Weak Lefschetz property, and let g be a Lefschetz element of A. We make the following observations about the Hilbert function and the socle type of A.

Remark 3.3. (1) Let d be the smallest degree for which $\times g : A_d \to A_{d+1}$ is surjective. Then the map $\times g : A_j \to A_{j+1}$ is also surjective for all $j \geqslant d$. This is because we are considering the natural grading.

(2) Hence $\times g : A_j \longrightarrow A_{j+1}$ is injective, but not surjective, for all $j < d$.

(3) Let $\underline{h} = (h_0, h_1, \ldots, h_s)$ be the Hilbert function of A. From (1) and (2) it follows that

$$h_0 < h_1 < \cdots < h_d \geqslant h_{d+1} \geqslant \cdots \geqslant h_s.$$

In particular, $\underline{h}$ is unimodal and strictly increasing until it reaches its peak, which is called the *Sperner number* of the Hilbert function of A [27].

(4) Thus we see that there exist integers $u_1, u_2, \ldots, u_\ell$ such that

$$h_0 < h_1 < \cdots < h_{u_1} = \cdots = h_{u_2-1} > h_{u_2} = \cdots = h_{u_3-1} > h_{u_3} \cdots > h_{u_\ell} = \cdots = h_s > 0.$$

In particular $u_1 = d$.

(5) Furthermore from (1) and (2) we have that the positive part of the first difference of $\underline{h}$, namely

$$1,\ h_1 - h_0,\ h_2 - h_1,\ \ldots,\ h_{u_1} - h_{u_1-1},$$

is the Hilbert function of $B = A/(g)$. In particular, this is an O-sequence.

(6) Let $(a_0, \ldots, a_s)$ be the h-vector of the socle of A. The Hilbert series of the socle is called the socle type $S(A, \lambda)$ of A, i.e.,

$$S(A, \lambda) = \sum_{i=0}^{s} a_i \lambda^i.$$

We want to compare the socle type with the following polynomial:

$$\Phi_{\underline{h}}(\lambda) := \sum_{i=u_1}^{s} (h_i - h_{i+1}) \lambda^i,$$

where $h_{s+1}=0$. It can easily be checked from (1), (2), and (4) that $a_i=0$ for all $i\notin\{u_2-1,u_3-1,\dots,u_\ell-1,s\}$. Furthermore, we have $a_i\leqslant h_i-h_{i+1}$ for all $i\in\{u_2-1,u_3-1,\dots,u_\ell-1,s\}$. This follows from

$$\operatorname{Soc}(A)_i\subset\ker(\times g: A_i\longrightarrow A_{i+1}),$$

$\dim\operatorname{Soc}(A)_i=a_i$, and $\dim\ker(\times g: A_i\longrightarrow A_{i+1})=h_i-h_{i+1}$. An Artinian K-algebra for which $a_i=h_i-h_{i-1}$ will be said to have *maximal socle type*. Notice that the rank of the last free module in the minimal free resolution of A is equal to $\sum a_i$, the dimension of the socle, so for an algebra with maximal socle type, this rank is actually equal to the Sperner number of A (see (3) above).

Conditions (3)–(5) give a necessary condition for a Hilbert function $\underline{h}$ to be the Hilbert function of an Artinian graded K-algebra with the Weak Lefschetz property. We now show that not only are these conditions also sufficient, thus characterizing the Hilbert functions of Artinian K-algebras with the Weak Lefschetz property, but in fact an example exists with the maximal possible socle type, as described in (6). We first give the basic construction.

Construction 3.4. Let $\underline{h}=(h_0,h_1,\dots,h_s,h_{s+1}=0)$ be a finite sequence of integers satisfying the conditions of (3)–(5) above. Define

$$\bar{h}(j):=\max\{h_j-h_{j-1},0\}.$$

Choose Artinian ideals

$$\overline{J}_1\subset\overline{J}_2\subset\cdots\subset\overline{J}_\ell\subset\overline{R}:=K[x_1,\dots,x_n]$$

such that $h_{\overline{R}/\overline{J}_1}=\bar{h}$ and $\deg\overline{J}_i=h(u_i)$ for all $i=2,\dots,\ell$. Now put $J_i=\overline{J}_iR$ for all $i=1,\dots,\ell$ and

$$I:=J_1+\sum_{i=2}^{\ell}[J_i]_{\geqslant u_i}+\mathfrak{m}^{s+1},$$

where $\mathfrak{m}=(x_0,\dots,x_n)$. Set $A:=R/I$. Note that J_i is not reduced, but it is the saturated ideal of a zero-scheme $\mathbb{X}_i$. Furthermore, we have $\mathbb{X}_1\supset\mathbb{X}_2\supset\cdots\supset\mathbb{X}_\ell$.

Proposition 3.5. *Let $\underline{h}=(1,h_1,\dots,h_s)$ be a finite sequence of positive integers. Then $\underline{h}$ is the Hilbert function of a graded Artinian K-algebra R/J having the Weak Lefschetz property if and only if $\underline{h}$ is a unimodal O-sequence such that the positive part of the first difference is an O-sequence.*

Furthermore, let $u_1,\dots,u_\ell$ and $\Phi_{\underline{h}}(\lambda)$ be as in Remark 3.3. *Then the K-algebra A of Construction* 3.4 *has the Weak Lefschetz property, Hilbert function $\underline{h}$ and maximal socle type $\Phi_{\underline{h}}(\lambda)$.*

Proof. The necessity is proved in Remark 3.3. The sufficiency follows immediately from the claim about Construction 3.4, which we now prove.

(1) The Artinian K-algebra A has the Weak Lefschetz property: Let $B^{(j)} := R/J_j = \bigoplus [B^{(j)}]_i$. We may assume that x_0 is not a zero divisor mod J_j for all j. Considering the following commutative diagram:

$$\begin{array}{ccccc} [B^{(j)}]_{u_{j+1}-1} & \xrightarrow{x_0} & [B^{(j)}]_{u_{j+1}} & \longrightarrow & [B^{(j+1)}]_{u_{j+1}} \\ \| & & & & \| \\ A_{u_{j+1}-1} & & \xrightarrow{\qquad x_0 \qquad} & & A_{u_{j+1}}, \end{array}$$

we have, as the proof of Lemma 3.2 in [13], that A has the Weak Lefschetz property.

(2) The Hilbert function of A is $\underline{h}$: First we recall a basic property of the Hilbert function of a zero-scheme $\mathbb{Y}$ in $\mathbb{P}^n$. Set

$$\sigma(\mathbb{Y}) := \min\bigl\{i \mid \Delta h_{R/I_{\mathbb{Y}}}(i) = 0\bigr\},$$

where $\Delta h_{R/I_{\mathbb{Y}}}(i)$ is the first difference of $h_{R/I_{\mathbb{Y}}}(i)$. Then it follows that

$$h_{R/I_{\mathbb{Y}}}(0) < \cdots < h_{R/I_{\mathbb{Y}}}\bigl(\sigma(\mathbb{Y}) - 1\bigr) = h_{R/I_{\mathbb{Y}}}\bigl(\sigma(\mathbb{Y})\bigr) = \cdots = \deg \mathbb{Y},$$

and we see that if $\mathbb{Y}' \subset \mathbb{Y}$ then $\sigma(\mathbb{Y}') \leqslant \sigma(\mathbb{Y})$. Hence from this property we get

$$h_{B^{(j)}}(i) = h_{u_j}$$

for all $i \geqslant u_1$. Thus since $A_i = [B^{(1)}]_i$ for all $0 \leqslant i \leqslant u_2 - 1$, $A_i = [B^{(j)}]_i$ for all $u_j \leqslant i \leqslant u_{j+1} - 1$ and $A_i = (0)$ for all $i \geqslant s + 1$, we have that the Hilbert function of A coincides with $\underline{h}$.

(3) The socle type of A is $\Phi_{\underline{h}}(\lambda)$: We note that

$$\bigl[\mathrm{Soc}(A)\bigr]_{u_{j+1}-1} = \bigl[I^{(j+1)}\bigr]_{u_{j+1}-1} \big/ \bigl[I^{(j)}\bigr]_{u_{j+1}-1}.$$

Furthermore we see that

$$\begin{aligned} \dim\bigl\{\bigl[I^{(j+1)}\bigr]_{u_{j+1}-1} \big/ \bigl[I^{(j)}\bigr]_{u_{j+1}-1}\bigr\} &= h_{B^{(j)}}(u_{j+1} - 1) - h_{B^{(j+1)}}(u_{j+1} - 1) \\ &= h_{u_j} - h_{u_{j+1}}. \end{aligned}$$

Thus it follows from Remark 3.3(6) that

$$S(A, \lambda) = \Phi_{\underline{h}}(\lambda).$$

This completes the proof. □

Example 3.6. Not all Artinian ideals in R whose Hilbert functions satisfy the necessary and sufficient conditions given in Proposition 3.5 have the Weak Lefschetz property. Indeed, we give a simple example of one which even has the Hilbert function of a complete intersection but does not have the Weak Lefschetz property. We take

$$I = \left(x_1^2, x_1x_2, x_1x_3, x_2^3, x_2^2x_3, x_2x_3^2, x_3^4\right),$$

so R/I has Hilbert function $(1\ 3\ 3\ 1)$. For any linear form L, the element $x_1 \in (R/I)_1$ is in the kernel of multiplication by L, hence the Weak Lefschetz property fails in passing from degree 1 to degree 2.

A finer invariant of an Artinian K-algebra is its minimal free resolution. It is probably not possible now to give a set of necessary and sufficient conditions on the graded Betti numbers for the existence of an ideal with the Weak Lefschetz property and that set of Betti numbers. Even in the Gorenstein case this is open. However, as in the Gorenstein case [19], we can give a sharp upper bound for the graded Betti numbers. We will do this shortly.

However, we begin with some natural questions, which are the analogs, for resolutions, of results which we know for Hilbert functions.

Question 3.7. (1) Is there a minimal free resolution (meaning only the graded Betti numbers, not the maps) corresponding to an Artinian ideal with a Hilbert function allowed by Proposition 3.5, which cannot occur for an ideal with the Weak Lefschetz property?

(2) Are there two Artinian ideals, I_1 and I_2, which have the same graded Betti numbers, but one has the Weak Lefschetz property and the other not?

We answer both of these questions. First we recall some terminology.

Definition 3.8. Let $>$ denote the degree-lexicographic order on monomial ideals, i.e., $x_1^{a_1}\cdots x_n^{a_n} > x_1^{b_1}\cdots x_n^{b_n}$ if the first nonzero coordinate of the vector

$$\left(\sum_{i=1}^{n}(a_i - b_i), a_1 - b_1, \dots, a_n - b_n\right)$$

is positive. Let J be a monomial ideal. Let m_1, m_2 be monomials in S of the same degree such that $m_1 > m_2$. Then J is a *lex-segment ideal* if $m_2 \in J$ implies $m_1 \in J$. When $\operatorname{char}(K) = 0$, we say that J is a *Borel-fixed ideal* if

$$m = x_1^{a_1}\cdots x_n^{a_n} \in J,\ a_i > 0, \quad \text{implies} \quad \frac{x_j}{x_i}\cdot m \in J$$

for all $1 \leqslant j < i \leqslant n$.

Example 3.9. We first answer Question 3.7(1). Let $J \subset K[x_1, x_2, x_3]$ be the lex-segment ideal for the Hilbert function $(1\ 3\ 3\ 1)$. Then its minimal free resolution is of the form

$$0 \longrightarrow \begin{matrix} R(-4) \\ \oplus \\ R(-5)^2 \\ \oplus \\ R(-6) \end{matrix} \longrightarrow \begin{matrix} R(-3)^3 \\ \oplus \\ R(-4)^5 \\ \oplus \\ R(-5)^2 \end{matrix} \longrightarrow \begin{matrix} R(-2)^3 \\ \oplus \\ R(-3)^3 \\ \oplus \\ R(-4) \end{matrix} \longrightarrow J \longrightarrow 0.$$

Now let I be any Artinian ideal in $K[x_1, x_2, x_3]$ with these graded Betti numbers. The generators of I in degree 2 have three linear syzygies. It is not hard to check (e.g., using methods of [3]) that this can only happen if they have a common linear factor (so in particular there is no regular sequence of length 2 among these three quadrics). But then after a change of variables we may assume that this common factor is x_1, and we are in the situation of Example 3.6. Hence R/I cannot have the Weak Lefschetz property. (As an alternative proof, note that the Socle type is $\lambda + 2\lambda^2 + \lambda^3$, so it also follows from Remark 3.3(6) that it cannot have the Weak Lefschetz property.)

Example 3.10. We now give a (positive) answer to Question 3.7(2). H. Ikeda has shown [16, Example 4.4] that there is a Gorenstein Artinian K-algebra $A = R/I$ with Hilbert function (1, 4, 10, 10, 4, 1) and minimal free resolution

$$0 \longrightarrow \mathbb{F}_4 \longrightarrow \mathbb{F}_3 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_1 \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where

$$\mathbb{F}_1 = R(-3)^{10} \oplus R(-4)^6, \qquad \mathbb{F}_2 = R(-4)^{15} \oplus R(-5)^{15},$$
$$\mathbb{F}_3 = R(-5)^6 \oplus R(-6)^{10}, \quad \text{and} \quad \mathbb{F}_4 = R(-9),$$

and not possessing the Weak Lefschetz property. These graded Betti numbers are precisely the maximum possible for this Hilbert function among ideals with the Weak Lefschetz property, and an ideal exists with these graded Betti numbers and with the Weak Lefschetz property, thanks to [19, Theorem 8.13].

In Example 3.9 we saw that the resolution of the lex-segment ideal (which is known to be extremal among all possible resolutions with the given Hilbert function [2,14,22] for $\operatorname{char} K > 0$) cannot, in general, be the minimal free resolution of an ideal with the Weak Lefschetz property, and we gave a reason for this failure based on the beginning of the resolution, and a different reason based on the end of the resolution. This suggests the following question:

Question 3.11. Let $\underline{h} = (h_0, h_1, \ldots, h_s)$ be a Hilbert function which can occur for Artinian K-algebras with the Weak Lefschetz property (see Proposition 3.5). Is there a maximal possible resolution among Artinian ideals with the Weak Lefschetz property and Hilbert function $\underline{h}$?

We now answer Question 3.11 by establishing upper bounds for the graded Betti numbers of an Artinian K-algebra with the Weak Lefschetz property and exhibiting examples where these bounds are attained. Note that such bounds were found for Artinian Gorenstein algebras with the Weak Lefschetz property in [19]. We adapt the techniques developed there to our problem.

We begin by recalling [19, Lemma 8.3].

Lemma 3.12. *Let M be a graded R-module, $\ell \in R$ a linear form. Then there is an exact sequence of graded R-modules* (*where* $\overline{R} := R/\ell R$)

$$\cdots \longrightarrow \operatorname{Tor}_{i-1}^{\overline{R}}\big((0:_M \ell), K\big)(-1) \longrightarrow \operatorname{Tor}_i^R(M, K) \longrightarrow \operatorname{Tor}_i^{\overline{R}}(M/\ell M, K) \longrightarrow \cdots$$
$$\longrightarrow \operatorname{Tor}_1^R(M, K) \longrightarrow \operatorname{Tor}_1^{\overline{R}}(M/\ell M, K) \longrightarrow 0.$$

Notation 3.13. Now let $A = R/I$ be an Artinian graded K-algebra with the Weak Lefschetz property, and let $g \in [R]_1$ be a Lefschetz element of A. Denote by d the end of A/gA and by a the initial degree of $0 : g := 0 :_A g$, i.e.,

$$d := \max\big\{ j \in \mathbb{Z} \mid [A/gA]_j \neq 0 \big\}$$
$$a := \min\big\{ j \in \mathbb{Z} \mid [0 : g]_j \neq 0 \big\}.$$

Observe that $d \leqslant a$. Using the notation of Remark 3.3 we have $d = u_1$, $a = u_2 - 1$.

Moreover, we put $\overline{R} := R/gR$ and define

$$\big[\operatorname{tor}_i^R(M, K)\big]_j := \operatorname{rank}\big[\operatorname{Tor}_i^R(M, K)\big]_j.$$

Now we can state the next result.

Proposition 3.14. *We have for all integers i, j:*

$$\big[\operatorname{tor}_i^R(A, K)\big]_{i+j} \begin{cases} = \big[\operatorname{tor}_i^{\overline{R}}(A/gA, K)\big]_{i+j} & \text{if } j \leqslant a-2; \\ \leqslant \big[\operatorname{tor}_i^{\overline{R}}(A/gA, K)\big]_{i+j} & \text{if } j = a-1; \\ \leqslant \big[\operatorname{tor}_{i-1}^{\overline{R}}(0 : g, K)\big]_{i+j-1} + \big[\operatorname{tor}_i^{\overline{R}}(A/gA, K)\big]_{i+j} & \text{if } a \leqslant j \leqslant d; \\ \leqslant \big[\operatorname{tor}_{i-1}^{\overline{R}}(0 : g, K)\big]_{i+j-1} & \text{if } j = d+1; \\ = \big[\operatorname{tor}_{i-1}^{\overline{R}}(0 : g, K)\big]_{i+j-1} & \text{if } j \geqslant d+2. \end{cases}$$

Furthermore, $\operatorname{Tor}_{n+1}^R(A, K) \cong \operatorname{Tor}_n^{\overline{R}}(0 : g, K)(-1)$.

Proof. Using $[\operatorname{Tor}_i^{\overline{R}}((0 :_A g), K)]_{i+j} = 0$ if $j < a$ and $[\operatorname{Tor}_i^{\overline{R}}(A/gA, K)]_{i+j} = 0$ if $j > d$, the claim follows by analyzing the exact sequence given in Lemma 3.12. □

Observe that the condition $a \leqslant j \leqslant d$ can only be satisfied if $a = d$.

Next, we need an elementary estimate.

Lemma 3.15. *Let M be a graded R-module. Then we have for all integers i, j:*

$$\left[\operatorname{tor}_i^R(M, K)\right]_{i+j} \leqslant h_M(j) \cdot \binom{n+1}{i}.$$

Proof. Put $P := R^{n+1}(-1)$. Then the Koszul complex gives the following minimal free resolution of $R/\mathfrak{m} \cong K$:

$$0 \longrightarrow \bigwedge^{n+1} P \longrightarrow \cdots \longrightarrow \bigwedge^{i+1} P \longrightarrow \bigwedge^{i} P \longrightarrow \cdots \longrightarrow P \longrightarrow R \longrightarrow K \longrightarrow 0.$$

Thus, $[\operatorname{Tor}_i^R(M, K)]_{i+j}$ is the homology of the complex

$$\left[\bigwedge^{i+1} P \otimes M\right]_{i+j} \longrightarrow \left[\bigwedge^{i} P \otimes M\right]_{i+j} \longrightarrow \left[\bigwedge^{i-1} P \otimes M\right]_{i+j}.$$

Since $\operatorname{rank}[\bigwedge^i P \otimes M]_{i+j} = h_M(j) \cdot \binom{n+1}{i}$, the claim follows. □

Notation 3.16. Let h be the Hilbert function of an Artinian K-algebra R/I. Then there is a uniquely determined lex-segment ideal $J \subset R$ such that R/J has h as its Hilbert function. We define

$$\beta_{i,j}(h, R) := \left[\operatorname{tor}_i^R(R/J, K)\right]_{i+j}.$$

Remark 3.17. The numbers $\beta_{i,j}(h, R)$ can be computed numerically without considering lex-segment ideals. Explicit formulas can be found in [10].

Theorem 3.18 [2,14,22]. *If $A = R/I$ is an Artinian algebra then we have for all $i, j \in \mathbb{Z}$:*

$$\left[\operatorname{tor}_i^R(A, K)\right]_{i+j} \leqslant \beta_{i,j}(h_A, R).$$

In order to construct algebras with the Weak Lefschetz property and maximal Betti numbers, we need one more technical result. In the following lemma, for a graded module M of finite length, we denote by $e(M)$ the last degree in which M is non-zero.

Lemma 3.19. *Let $\bar{I} \subset \overline{J} \subset \overline{R} := K[x_1, \ldots, x_n]$ be Artinian ideals. Put $d := e(\overline{R}/\bar{I})$, $I = \bar{I}R$, $J := \overline{J}R$, and $\mathfrak{a} := I + [J]_{\geqslant d+1}$. Then $\mathfrak{a} + x_0R = I + x_0R$ and we have for the graded Betti numbers of $A := R/\mathfrak{a}$:*

$$\left[\operatorname{tor}_i^R(A, K)\right]_j = \begin{cases} \left[\operatorname{tor}_i^{\overline{R}}(A/x_0A, K)\right]_j & \text{if } j \neq i + d; \\ \left[\operatorname{tor}_i^{\overline{R}}(A/x_0A, K)\right]_j + k \cdot \binom{n}{i-1} & \text{if } j = i + d, \end{cases}$$

where $k := \deg I - \deg J$.

Proof. We proceed in several steps.

(I) Since $\bar{I} \subset \overline{J}$, we get $e(\overline{R}/\overline{J}) \leqslant e(\overline{R}/\bar{I}) = d$. Hence, $\bar{I}$ and $\overline{J}$ are generated by forms of degree $\leqslant d+1$. In particular, $[J]_{\geqslant d+1}$ is generated by forms of degree $d+1$.

The ideals $I + x_0 R$ and $\mathfrak{a} + x_0 R$ differ at most in degrees $\geqslant d+1$. Thus, the Hilbert functions of $A/x_0 A$ and $\overline{R}/\bar{I}$ agree. It follows that $I + x_0 R = \mathfrak{a} + x_0 R$. In particular, we can write

$$\mathfrak{a} = I + x_0 \cdot (f_1, \ldots, f_k),$$

where $f_1, \ldots, f_k \in [J]_d$, because $J : x_0 = J$.

(II) Put $\mathfrak{b} := (f_1, \ldots, f_k)R$, i.e., $\mathfrak{a} = I + x_0 \cdot \mathfrak{b}$. For $j \leqslant d$, multiplication by x_0 factors through two maps of maximal rank:

$$\begin{array}{ccccc} [A]_j & & \xrightarrow{\quad x_0 \quad} & & [A]_{j+1} \\ \| & & & & \| \\ [R/I]_j & \xrightarrow{x_0} & [R/I]_{j+1} & \longrightarrow & [R/\mathfrak{a}]_{j+1}. \end{array}$$

It follows that

$$0 :_A x_0 \cong [\mathfrak{a}/I]_d \cong K^k(-d)$$

and, in particular, $0 :_A x_0 \cong \operatorname{Soc} A$.

(III) Denote by $g_1, \ldots, g_t$ the minimal generators of I. Let $(r_1, \ldots, r_t, s_1, \ldots, s_k)^{\mathrm{t}}$ be a syzygy of $\mathfrak{a}$, i.e.,

$$\sum_{i=1}^{t} r_i g_i + \sum_{j=1}^{k} s_j x_0 f_j = 0.$$

We can write $r_i = \bar{r}_i + x_0 \tilde{r}_i$ where $\bar{r}_i \in \overline{R}$ and $\tilde{r}_i \in R$. It follows that

$$\sum_{i=1}^{t} \bar{r}_i g_i + x_0 \left[\sum_{i=1}^{t} \tilde{r}_i g_i + \sum_{j=1}^{k} s_j f_j \right] = 0.$$

Comparing coefficients we obtain $\sum_{i=1}^{t} \bar{r}_i g_i = 0$ and $\sum_{i=1}^{t} \tilde{r}_i g_i + \sum_{i=1}^{k} s_j f_j = 0$. Thus, we see that $(\bar{r}_1, \ldots, \bar{r}_t, 0, \ldots, 0)^{\mathrm{t}} + (x_0 \tilde{r}_1, \ldots, x_0 \tilde{r}_t, s_1, \ldots, s_k)^{\mathrm{t}}$ is a syzygy of $\mathfrak{a}$ if and only if $(\bar{r}_1, \ldots, \bar{r}_t)^{\mathrm{t}}$ is a syzygy of I and $(\tilde{r}_1, \ldots, \tilde{r}_t, s_1, \ldots, s_k)^{\mathrm{t}}$ is a syzygy of $I + \mathfrak{b}$.

(IV) Let

$$0 \longrightarrow \overline{G}_n \longrightarrow \cdots \longrightarrow \overline{G}_2 \xrightarrow{\bar{\alpha}} \overline{G}_1 \xrightarrow{\bar{\beta}} \overline{R} \longrightarrow \overline{R}/\bar{I} \longrightarrow 0$$

be a minimal free resolution of $\overline{R}/\bar{I}$ as $\overline{R}$-module and let

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow A \longrightarrow 0$$

be a minimal free resolution of A as R-module. Tensoring by $\overline{R}$ gives the complex (with $\overline{F}_i := F_i \otimes_R \overline{R}$)

$$0 \longrightarrow \overline{F}_n \longrightarrow \cdots \longrightarrow \overline{F}_2 \xrightarrow{\alpha} \overline{F}_1 \xrightarrow{\beta} \overline{F}_0 \longrightarrow R/\mathfrak{a} + x_0 R \longrightarrow 0.$$

Since $\mathfrak{a} + x_0 R = I + x_0 R$, we get

$$\ker\beta \cong \ker\bar{\beta} \oplus \overline{R}^k(-d-1).$$

Step (III) shows that $\operatorname{im}\alpha$ splits as

$$\operatorname{im}\alpha \cong \operatorname{im}\bar{\alpha} \oplus M \tag{*}$$

for some $\overline{R}$-module M such that

$$\ker\beta/\operatorname{im}\alpha \cong \overline{R}^k(-d-1)/M.$$

The proof of [19, Lemma 8.3] shows that $\ker\beta/\operatorname{im}\alpha \cong 0 :_A x_0(-1)$. Using step (II) we obtain the exact sequence of $\overline{R}$-modules

$$0 \longrightarrow M \longrightarrow \overline{R}^k(-d-1) \longrightarrow K^k(-d-1) \longrightarrow 0.$$

It implies for all integers $i \geqslant 0$:

$$\operatorname{Tor}_i^{\overline{R}}(M, K) \cong K^{k\binom{n}{i+1}}(-d-2-i).$$

From the proof of [19, Lemma 8.3] we also have for $i \geqslant 0$:

$$\operatorname{Tor}_{i+2}^{R}(A, K) \cong \operatorname{Tor}_i^{\overline{R}}(\operatorname{im}\alpha, K).$$

Hence, the sequence $(*)$ implies our claim. □

We are now ready for the announced result.

Theorem 3.20.

(a) *Let $A = R/I$ be a K-algebra with the Weak Lefschetz property and denote by $\bar{h}: \mathbb{Z} \to \mathbb{Z}$ the function defined by*

$$\bar{h}(j) := \max\{\Delta h_A(j), 0\}.$$

Then the graded Betti numbers of A satisfy

$$\big[\operatorname{tor}_i^R(A,K)\big]_{i+j} \leqslant \begin{cases} \beta_{i,j}\big(\bar h, \overline{R}\big) & \text{if } j \leqslant a-1; \\ \beta_{i,j}\big(\bar h, \overline{R}\big) + \max\{0, -\Delta h_A(j+1)\} \cdot \binom{n}{i-1} & \text{if } a \leqslant j \leqslant d; \\ \max\{0, -\Delta h_A(j+1)\} \cdot \binom{n}{i-1} & \text{if } j \geqslant d+1. \end{cases}$$

(b) *Let $h:\mathbb{Z} \to \mathbb{Z}$ be a numerical function such that there is an Artinian algebra R/J having the Weak Lefschetz property and h as a Hilbert function. Then there is an Artinian algebra $A = R/I$ having the Weak Lefschetz property and h as a Hilbert function such that equality is true in* (a) *for all integers i, j.*

Proof. We first prove (a). Since g is a Lefschetz element of A, the Hilbert function of A/gA is $\bar h$ and the Hilbert function of $0 :_A g$ is given by

$$h_{0:_A g}(j) = \max\big\{0, -\Delta h_A(j+1)\big\}.$$

Thus, our claim is a consequence of Proposition 3.14, Lemma 3.15, and Theorem 3.18 (using [22] for the case char $K > 0$).

Now we show (b). We use the notation of Remark 3.3. Consider the ideal I of Construction 3.4, and assume furthermore that

$$\big[\operatorname{tor}_i^{\overline{R}}\big(\overline{R}/\overline{J}_1, K\big)\big]_{i+j} = \beta_{i,j}\big(\bar h, \overline{R}\big) \quad \text{for all integers } i, j.$$

Such an ideal $\overline{J}_1$ certainly exists: for example, we can choose it as a lex-segment ideal.

As in step (I) of the proof of Lemma 3.19 we see that $I + x_0R = J_1 + x_0R$. An argument as in step (II) of that proof shows that

$$0 :_A x_0 = \operatorname{Soc} A \quad \text{and} \quad \operatorname{rank}[0 :_A x_0]_j = \max\big\{0, -\Delta h(j+1)\big\}.$$

It follows that A has the Weak Lefschetz property, x_0 is a Lefschetz element for A and

$$\big[\operatorname{tor}_i^{\overline{R}}(0 :_A x_0, K)\big]_{i+j} = \max\big\{0, -\Delta h(j+1)\big\} \cdot \binom{n}{i}.$$

Moreover, since $A/x_0A \cong \overline{R}/\overline{J}_1$, we have

$$\big[\operatorname{tor}_i^{\overline{R}}(A/x_0A, K)\big]_{i+j} = \beta_{i,j}\big(\bar h, \overline{R}\big).$$

Observe again that $d = u_1$ and $a := u_2 - 1 \geqslant d$. If $a \geqslant d+1$, all Betti numbers $[\operatorname{tor}_i^R(A,K)]_{i+j}$ are determined by Proposition 3.14 if $j \geqslant d+2$. Since $[A]_j = [R/J_1]_j$, for $j \leqslant a$ we get

$$\big[\operatorname{tor}_i^R(A,K)\big]_{i+j} = \big[\operatorname{tor}_i^R(R/J_1,K)\big]_{i+j} = \big[\operatorname{tor}_i^{\overline{R}}(A/x_0A,K)\big]_{i+j} \quad \text{if } j \leqslant d.$$

The remaining graded Betti numbers $[\operatorname{tor}_i^R(A, K)]_{i+d+1}$ can be computed recursively from the Hilbert function of A. (A similar computation can be found in [21, p. 4386].)

Now let $a = d$. From the definition of I we immediately obtain

$$\big[\operatorname{tor}_i^R(A, K)\big]_{i+j} = \big[\operatorname{tor}_i^R\big(R/(J_1 + [J_2]_{\geqslant a}), K\big)\big]_{i+j} \quad \text{for all } j \leqslant d.$$

Thus, we know these graded Betti numbers by Lemma 3.19. If $j \geqslant d + 2$, we know $[\operatorname{tor}_i^R(A, K)]_{i+j}$ by Proposition 3.14. Thus, the remaining Betti numbers can be computed as in the previous case.

In any case, we can compute all graded Betti numbers of A. The result shows our claim. □

We would also like to point out that there are Hilbert functions such that all algebras with that Hilbert function and the Weak Lefschetz property have the same (maximal) graded Betti numbers. A similar phenomenon is true for Gorenstein algebras with the Weak Lefschetz property (cf. [19, Corollary 8.14]).

Corollary 3.21. *Let $I \subset R$ be an Artinian ideal such that $A := R/I$ has the Weak Lefschetz property and its Hilbert function satisfies*

$$h_A(j) = \binom{n+j}{n} \quad \textit{for all } j \leqslant d = u_1 \leqslant u_2 - 3$$

and $u_k + 2 \leqslant u_{k+1}$ for all k with $2 \leqslant k < \ell$. Then the graded Betti numbers of A are

$$\big[\operatorname{tor}_i^R(A, K)\big]_{i+j} = \begin{cases} \binom{n+d}{i+d} & \textit{if } j = d; \\ -\Delta h_A(u_k) \cdot \binom{n}{i-1} & \textit{if } j = u_k - 1; \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. By assumption we have $a \geqslant d + 2$. Thus, Lemma 3.12 provides

$$\big[\operatorname{tor}_i^R(A, K)\big]_{i+j} = \begin{cases} \big[\operatorname{tor}_i^{\overline{R}}(A/gA, K)\big]_{i+j} & \text{if } j \leqslant d; \\ 0 & \text{if } j = d+1; \\ \big[\operatorname{tor}_{i-1}^{\overline{R}}(0 : g, K)\big]_{i+j-1} & \text{if } j \geqslant d+2. \end{cases}$$

We may assume that $g = x_0$. Then we get $A/x_0A \cong \overline{R}/(x_1, \ldots, x_n)^{d+1}$. Thus, the graded Betti numbers of A/gA are known (cf., e.g., the proof of [19, Corollary 8.14]). This shows our claim for $j \leqslant d + 1$.

Since A has the Weak Lefschetz property, we have

$$\operatorname{rank}[0 :_A x_0]_j = \max\big\{0, -\Delta h(j+1)\big\}.$$

This implies

$$0 :_A x_0 = \operatorname{Soc} A.$$

Our claim follows. □

4. Hilbert functions and maximal Betti numbers of algebras with the Strong Lefschetz property

In this section we give some results about a more stringent condition, namely the Strong Lefschetz property. Several of our results require that $\operatorname{char} K = 0$ (e.g., Proposition 4.4), and we make this assumption throughout this section.

Not all algebras with the Weak Lefschetz property possess the Strong Lefschetz property in codimension $\geqslant 3$. We show that nevertheless this *does* hold in codimension two. Furthermore, we give the surprising result that the *same* characterization of Hilbert functions and maximal graded Betti numbers that we gave in the last section for the Weak Lefschetz property continues to hold for the Strong Lefschetz property.

The conditions for the Hilbert function given in the statement of Proposition 3.5 are automatic in codimension two. In this case, interestingly, something much stronger than Proposition 3.5 holds. We first recall the notion of the Strong Lefschetz property.

Definition 4.1. An Artinian ideal $I \subset R$ has the *Strong Lefschetz property* if, for a general linear form L and any $d > 0, i \geqslant 0$, the map

$$\times L^d : (R/I)_i \to (R/I)_{i+d}$$

has maximal rank.

Clearly if R/I has the Strong Lefschetz property then it has the Weak Lefschetz property. However, there are examples of ideals with the Weak Lefschetz property which do not have the Strong Lefschetz property.

Example 4.2. We first give a simple example of an ideal with the Weak Lefschetz property but not the Strong Lefschetz property. Let I be the lex-segment ideal with generators

$$x_1^2,\ x_1x_2,\ x_1x_3^2,\ x_2^3,\ x_2^2x_3^2,\ x_2x_3^3,\ x_3^5.$$

This has Hilbert function (1 3 4 3 1), and one can check that for multiplication by a general linear form L we have maximal rank between consecutive components, while L^2 has the element x_1 in the kernel of the multiplication from degree 1 to degree 3.

Of much greater interest is the fact that there exist examples of *Gorenstein* ideals with the Weak Lefschetz property but not the Strong Lefschetz property. One uses the theory of inverse systems.

Example 4.3. Let R be the ring $K[u, v, x, y, z]$ and let $f = xu^2 + yuv + zv^2$. The dual of f gives a Gorenstein algebra with h-vector (1 5 5 1) (this can be checked, for instance, with the computer program Macaulay [1] using the script `<l_from_dual`). This algebra has neither the Weak Lefschetz property nor the Strong Lefschetz property.

However, now take the polynomial $g = uf$. It gives an algebra with h-vector (1 5 6 5 1). It has the Weak Lefschetz property but *not* the Strong Lefschetz property.

More generally, choose an element $g \in S = [u, v][f]$ which is, in particular, homogeneous in the variables x, y, z, u, v. Let A be the algebra obtained from such a form. Then for a general linear form L, the map $\times L^{s-2}: A_1 \to A_{s-1}$ is not bijective. The key to this goes back to P. Gordan and M. Noether [12]. They showed that if the Hessian of a form is identically zero then one of the variables can be eliminated by means of a linear change of the variables, as long as the number of variable is at most *four*. In dimension 5 or more it is not true, and they gave the above example. In dimension 5 they claimed that these types of forms are the only cases, where you have zero Hessian and still all variables are essentially involved. Then the fourth author [29] showed that the zero Hessian of a form is equivalent to the condition that the map $g^{s-2}: A_1 \to A_{s-1}$ does not have full rank.

We believe that in general a polynomial of the above form does give rise to an Artinian algebra with the Weak Lefschetz property, but have not confirmed it.

We saw in Example 3.6 that for a given Hilbert function in codimension $\geqslant 3$ it is possible to find two ideals with that Hilbert function, one possessing the Weak Lefschetz property and the other not. In codimension two we have the following proposition, generalizing some results in [15].

Proposition 4.4. *Every Artinian ideal in* $K[x, y]$ $(\mathrm{char} K = 0)$ *has the Strong Lefschetz property* (*and consequently also the Weak Lefschetz property*).

Proof. First suppose that I is a Borel-fixed ideal in $R = K[x, y]$. Since $\mathrm{char}\, K = 0$, I_d consists of consecutive monomials from the first (each d). (Say x^d is the first monomial and y^d the last.) So the vector space R/I_d is spanned by the consecutive monomials from the last.

Let $(h_0, h_1, \dots, h_s)$ be the Hilbert function of $A = R/I$. Then it is well known (and easy to see) that it is unimodal. Assume first that $h_i \leqslant h_{i+d}$. Then $y^d: (R/I)_i \to (R/I)_{i+d}$ is injective, because if a monomial M is in $(R/I)_i$ then $y^d M$ is in $(R/I)_{i+d}$. (The point here is that if M is the tth monomial of $(R/I)_i$ from the last then $y^d M$ is also the tth monomial of $(R/I)_{i+d}$ from the last.)

Now assume that $h_i \geqslant h_{i+d}$. Suppose that a monomial M is in $(R/I)_{i+d}$. Say M is the tth monomial from the last. Then the tth monomial of $(R/I)_i$ from the last exists since $h_i > h_{i+d}$. Let it be N. Then we have $y^d N = M$. Thus the map $y^d: (R/I)_i \to (R/I)_{i+d}$ is surjective. Hence we have proved that if I is Borel-fixed in characteristic 0, then R/I has the Strong Lefschetz property.

In the general case we have the fact that $\mathrm{gin}(I)$ is Borel-fixed, where $\mathrm{gin}(I)$ denotes the generic initial ideal of I. It is easy to see and well known (or see Proposition 15.12 of Eisenbud [9]) that $\mathrm{In}(I : y^d) = \mathrm{In}(I) : y^d$ for $d = 1, 2, 3 \dots$, where y is the last variable with respect to the graded reverse lexicographic order. Since the Hilbert function does not change by passing to $\mathrm{gin}(I)$ and since the Strong Lefschetz property is characterized by the Hilbert function of $A/(y^d)$, $d = 1, 2, 3 \dots$, the general case reduces to the case of Borel-fixed ideals. □

We have seen that the Strong Lefschetz property is (naturally) a stronger condition than the Weak Lefschetz property, in the sense that there exist ideals whose coordinate ring has the Weak Lefschetz property but not the Strong Lefschetz property. One would naturally expect that the imposition of this extra condition would be accompanied by a further restriction on the possible Hilbert functions (Proposition 3.5) or on the upper bounds on the graded Betti numbers (Theorem 3.20).

We now show that any Hilbert function that occurs for ideals with the Weak Lefschetz property also occurs for ideals with the Strong Lefschetz property. The following two results do not require char $K = 0$.

Proposition 4.5. *Let K be any field. Let I be the ideal obtained in Construction* 3.4*, with the further assumption that $\overline{J}_2, \ldots, \overline{J}_\ell$ satisfy*

$$h_{\overline{R}/\overline{J}_i}(t) = \Delta h^{(i)}(t) \quad \text{for all } i = 2, \ldots, \ell, \quad \text{where } h^{(i)}(t) := \begin{cases} \min\{h_t, h_{u_i}\} & \text{if } t < u_i, \\ h_{u_i} & \text{otherwise.} \end{cases}$$

(*Such ideals certainly exist. For example, we can choose those as lex-segment ideals.*) *Then $A = R/I$ has the Strong Lefschetz property.*

Proof. We maintain the notation of Construction 3.4 and Proposition 3.5. We may assume that x_0 is not a zero divisor mod J_j. First suppose that $i + d < u_2$. Then from the proof of Proposition 3.5, we see that (A, x_0) has the Weak Lefschetz property. Hence it follows that the map $\times x_0^d : A_i \longrightarrow A_{i+d}$ is injective.

So without loss of generality we may assume that $u_j \leqslant i + d \leqslant u_{j+1} - 1$ (where $2 \leqslant j \leqslant \ell$ and $u_{\ell+1} := s + 1$). We note that

$$h_{B^{(j)}}(t) = \begin{cases} h_t & \text{if } 0 \leqslant t \leqslant \sigma(\mathbb{X}_j) - 2; \\ h_{u_j} & \text{otherwise.} \end{cases}$$

Hence we see that

$$\text{the natural map } A_i \longrightarrow B_i^{(j)} \text{ is } \begin{cases} \text{bijective} & \text{if } 0 \leqslant i \leqslant \sigma(\mathbb{X}_j) - 2; \\ \text{surjective} & \text{if } \sigma(\mathbb{X}_j) - 1 \leqslant i \leqslant u_j - 1; \\ \text{bijective} & \text{if } u_j \leqslant i \leqslant u_{j+1} - 1. \end{cases}$$

Also we note that

$$x_0^d : B_i^{(j)} \longrightarrow B_{i+d}^{(j)} \text{ is } \begin{cases} \text{injective} & \text{if } i \leqslant \sigma(\mathbb{X}_j) - 2; \\ \text{bijective} & \text{otherwise.} \end{cases}$$

Thus, considering the following commutative diagram

$$\begin{array}{ccc} A_i & \xrightarrow{x_0^d} & A_{i+d} \\ \downarrow & & \| \\ B_i^{(j)} & \xrightarrow{x_0^d} & B_{i+d}^{(j)}, \end{array}$$

we have

$$x_0^d : A_i \longrightarrow A_{i+d} \text{ is } \begin{cases} \text{injective} & \text{if } i \leqslant \sigma(\mathbb{X}_j) - 2; \\ \text{surjective} & \text{otherwise.} \end{cases} \qquad \square$$

The next result shows that the bounds on the graded Betti numbers that were given in Theorem 3.20 are also achieved by an ideal with the Strong Lefschetz property.

Corollary 4.6. *Let K be any field. A Hilbert function $\underline{h}$ occurs for some graded Artinian K-algebra with the Weak Lefschetz property if and only if it occurs for one with the Strong Lefschetz property, and these Hilbert functions are characterized in Proposition* 3.5. *Furthermore, the bound on the graded Betti numbers obtained in Theorem* 3.20 *is achieved by an algebra with the Strong Lefschetz property.*

Proof. The only thing that needs to be observed is that the extra condition on $\overline{J}_1$ imposed in Theorem 3.20, namely

$$\left[\operatorname{tor}_i^{\overline{R}}(\overline{R}/\overline{J}_1, K)\right]_{i+j} = \beta_{i,j}\left(\bar{h}, \overline{R}\right) \qquad \text{for all integers } i, j,$$

can be imposed in the context of Proposition 4.5: simply take $\overline{J}_1$ to be a lex-segment ideal. □

We end with a natural question.

Question 4.7. Is there a set of graded Betti numbers that occurs for algebras with the Weak Lefschetz property but not the Strong Lefschetz property?

We conjecture the answer to this question to be "no."

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LIST OF PROBLEMS

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Abstract. The study of the Lefschetz properties of Artinian graded algebras was motivated by the hard Lefschetz theorem for a smooth complex projective variety, a breakthrough in algebraic topology and geometry. Over the last few years, this topic has attracted increasing attention from mathematicians in various areas. Here, we suggest some important open problems about or related to Lefschetz properties of Artinian graded algebras with the ultimate aim to attract the attention of young researchers from different areas.

1. Introduction

As it was pointed out in the previous chapters of this book, the study of Lefschetz properties of Artinian algebras was originally motivated by the Lefschetz theory for projective manifolds, begun by S. Lefschetz and well-established by the late 1950s. Many of the important Artinian graded algebras appear as cohomology rings of an algebraic variety or manifold, though recent important developments have demonstrated cases of the Lefschetz property beyond such geometric settings (such as Coxeter groups or matroids). Lefschetz properties also appear as one important ingredient of the Kähler package. In the last two decades there has been fascinating progress on the study of the weak and strong Lefschetz property from different perspectives, inspired in part by, and contributing to, developments in algebraic geometry, commutative algebra and combinatorics among others but we want to emphasize that in spite of the big progress in the area made during these last decades a lot of interesting problems remain open. Lefschetz properties have shown to be ubiquitous since this subject matter has connections to many branches of mathematics. Indeed, a central object of study are Gorenstein algebras (also known as Poincaré duality algebras) which are of strong interest not only in algebraic geometry but also in commutative algebra, algebraic topology and combinatorics. Notably, several important results in this area have been obtained by using unexpected methods and finding unexpected connections between apparently different topics.

In this last chapter, we gather a collection of open problems about or related to Lefschetz properties with the ultimate aim to attract the attention of young researchers from different areas. Many researchers have contributed to this list which in no means claims to be an exhaustive list but which hopefully gives a flavor of the problems in this area. Moreover, the suggested problems vary significantly in their level of detail.

In the following, we divide the problems into 4 blocks:

(1) Failure/presence of Lefschetz properties for special types of algebra.
(2) Geometric aspects related to Lefschetz properties.

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(3) Lefschetz properties in combinatorics.
(4) Lefschetz properties and Jordan types of Artinian algebras.

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2. Failure/presence of Lefschetz properties for special types of algebra

Though many algebras are expected to have the weak Lefschetz property (WLP, for short), establishing this property is often rather difficult. In this first section we gather some open problems related to the failure/presence of the WLP for special types of Artinian algebras. We also mention that some of the main results that have been achieved up to date only hold over fields of characteristic zero or under other restrictions on the characteristic of the ground field. It is hence of great interest to understand the influence of the characteristic of the ground field on Lefschetz type questions. If not otherwise stated, in this section, we will always assume to work over a field $\mathbb{K}$ of characteristic zero.

Complete Intersections. It is known that all graded algebras in two or fewer variables have the WLP [24, Proposition 4.4]. In particular, these includes height two complete intersections in two or fewer variables. Following up on our previous comment on the characteristic dependence, we want to mention that it is not hard to see that complete intersections in two variables can fail the WLP in positive characteristic. Coming back to characteristic zero, Harima, Migliore, Nagel and Watanabe [24] showed that any Artinian height three complete intersection has the WLP. On the other hand, Stanley [45] and Watanabe [47] showed that *any* monomial complete intersection, independent of the number of variables, has even the strong Lefschetz property (SLP, for short) (and hence the WLP). From this, the following question arises naturally:

Question 2.1. *Does every complete intersection ideal $I \subseteq \mathbb{K}[x_1, \dots, x_n]$, in any number of variables, has the WLP (or even the SLP)?*

This question should be considered as the main open problem concerning Lefschetz properties for complete intersections. In a step towards an answer to Question 2.1, it seems reasonable to first try to extend the work in [24] as follows.

Question 2.2. *Does every height three complete intersection has the SLP? Is this true for complete intersections of larger height, e.g., Artinian height four?*

Even, more specific is the following problem, whose answer is also unknown.

Question 2.3. *Let $I \subseteq \mathbb{K}[x_1, \dots, x_n]$ be a complete intersection generated by homogeneous polynomials of uniform degree $d \in \mathbb{N}$. Further assume that $\mathbb{K}[x_1, \dots, x_n]/I$ has the same monomial basis as $\mathbb{K}[x_1, \dots, x_n]/(x_1^d, x_2^d, \dots, x_n^d)$. Does $\mathbb{K}[x_1, \dots, x_n]/I$ satisfy the SLP?*

The last problem we are proposing for complete intersections is concerned with Lefschetz properties for non-standard graded Artinian complete intersection and relates to an old conjecture by Almkvist [7, 8, 9] which we now explain. Let $p_{n,d}(t)$ be the generating function for the number of partitions of integers smaller than or equal to $(d-1)n(n+1)/2$ with at most n parts and each part repeated at most $d-1$ times. It is known that

$$p_{n,d}(t) = \prod_{i=1}^{n} \left(1 + t^i + \cdots + t^{i(d-1)}\right).$$

Conjecture 2.4 (Almkvist '85). *For every fixed $d \geq 2$, the polynomial $p_{n,d}(t)$ has unimodal coefficients for n sufficiently large.*

Almkvist has already established his conjecture [9] for $3 \leq d \leq 20$ (and also, oddly enough, for $d = 100$ and $d = 101$) using analytic techniques, including integrals and Tchebychev polynomials.

One possible approach towards Conjecture 2.4 is via Lefschetz properties. We now explain this in more detail. Given integers $n, d \geq 1$, let $e_i = e_i(x_1, \ldots, x_n)$ be the i^{th} *elementary symmetric polynomial* in the variables $x_1, \ldots, x_n$, and let $\hat{e}_i = e_i(x_1^d, \ldots, x_n^d)$ be the i^{th} elementary symmetric polynomials in the variables raised to the d^{th} powers $x_1^d, \ldots, x_n^d$. It is known that $\mathbb{K}[e_1, \ldots, e_n]$ is a polynomial ring, and that $\{\hat{e}_1, \ldots, \hat{e}_n\} \subseteq \mathbb{K}[e_1, \ldots, e_n]$ form a regular sequence. Consequently, the quotient ring $A(n,d) = \frac{\mathbb{K}[e_1,\ldots,e_n]}{(\hat{e}_1,\ldots,\hat{e}_n)}$ is a (non-standard) graded Artinian complete intersection. It is easy to see that the Poincaré polynomial of $A(n,d)$ equals $p_{n,d}(t)$. Hence, an answer to the following question would solve Conjecture 2.4.

Question 2.5. *For which pairs (n,d) does the ring $A(n,d)$ have the SLP?*

Gorenstein algebras. The question of which Gorenstein algebras do possess the WLP (or even the SLP) is even more mysterious than the corresponding one for complete intersections. On one hand, as mentioned above, all graded algebras in two or fewer variables have the WLP [24, Proposition 4.4]. On the other hand, Ikeda and Boij showed that there *exist* Artinian Gorenstein algebras of codimension > 3 failing the WLP. In view of the fact, that complete intersections of codimension 3 do have the WLP, it is natural to ask the following question:

Question 2.6. *Do all Artinian codimension three Gorenstein algebras have the WLP (or even the SLP)? If not, can one characterize those which do?*

Building on work of Boij and Ikeda, one might also aim to characterize (or at least better understand) which Gorenstein algebras do have the WLP/SLP, e.g., in terms of properties of their Hilbert functions. However, it is not enough to only consider Gorenstein algebras with unimodal Hilbert functions (which is a consequence of the WLP), since, even in this more restricted setting, there exist examples of Gorenstein algebras that do not have the WLP.

As for complete intersections, we now mention a more specific problem.

Question 2.7. *Let $A = \bigoplus_{i=0}^{c} A_i$ be a standard graded Artinian Gorenstein algebra over a field of characteristic zero or greater than c with embedding dimension n. Suppose that the symmetric group S_n acts on A by permutation of the variables. Does A have the SLP?*

The above question has been answered in the affirmative for quadratic complete intersections.

Jacobian ideals. Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables with coefficients in $\mathbb{C}$ and let $f \in S$ be a homogeneous polynomial of degree d whose corresponding hypersurface $V = V(f) = \{x \in \mathbb{P}^n \ : \ f(x) = 0\}$ in $\mathbb{P}^n$ is smooth. Let J_f be the *Jacobian ideal* of f, generated by the partial derivatives f_i of f with respect to x_i for $i = 0, \dots, n$. The graded algebra $M(f) = S/J_f$ is called the *graded Milnor* (or *Jacobian*) algebra of f. If f is generic, it follows from [45] and [47] that $M(f)$ has both the WLP and the SLP. Moreover, for any $f \in \mathbb{C}[x_0, \dots, x_3]$ of degree $d \leq 6$ with $V(f)$ smooth we know that $M(f)$ satisfies the WLP (see [32] and [13]). This gives rise to the following question:

Question 2.8. *Does $M(f)$ have the WLP (or the SLP) for any homogeneous form $f \in \mathbb{C}[x_0, \dots, x_n]$ with $V(f) \subset \mathbb{P}^n$ smooth?*

Monomial algebras. Though it is out of reach to provide a complete classification of all Artinian monomial algebras A that have the WLP, one can still ask for a characterization for special classes. Two natural subclasses come from looking at the end and at the beginning of a minimal free resolution of A.

An algebra A is called *level* if the last free module in its minimal free resolution is concentrated in one degree. The rank of the last free module is called the *type* of A. It is natural to ask which monomial level algebras of a prescribed type have the WLP. It was shown in [45] and [47] that for type 1 this is always true. Moreover, in [12] it was proven that also monomial level algebras of type 2 in three variables always have the WLP, whereas for every other number of variables (≥ 3) and every possible type there exist monomial level algebras not having the WLP. This motivates the following question:

Problem 2.9. *Let n and t be positive integers. Find or bound the minimal degree d such that there exists a monomial level algebra in n variables of type t whose last syszygy module is generated in degree d and fails the WLP.*

We want to emphasize that obviously this question also makes sense for the SLP, for which even less is known in this setting.

Another class of monomial algebras, which is natural to consider are *monomial almost complete intersections*. For these, several authors have studied the special case of three variables. It would be interesting to know what happens if we allow one more generator. It is also not known what happens if the algebra is not required to be monomial but just an almost complete intersection.

Once more, the results that are known in characteristic zero change dramatically in characteristic p. A broad question is the following:

Question 2.10. *Let I be a monomial ideal such that R/I has the WLP in characteristic zero. What are the field characteristics in which R/I fails to have the WLP?*

Powers of linear forms. Many other natural algebras lend themselves to questions about the WLP or the SLP. A popular instance is that the underlying ideal is generated by powers of linear forms (see, for instance, [33]). Most of the results achieved in this direction rely on a result of

Emsalem and Iarrobino [21], that translates the problem of whether the considered algebra has the WLP to one of studying sets of fat points in a projective space. In [37], the following conjecture was stated:

Conjecture 2.11. *Let $R = \mathbb{K}[x_1, \dots, x_{2n+1}]$. Let $L \in R$ be a general linear form, and let $I = \langle x_1^d, \dots, x_{2n+1}^d, L^d \rangle$.*

(i) *If $n = 3$, the algebra R/I fails the WLP if and only if $d \geq 3$.*
(ii) *If $n \geq 4$, the algebra R/I fails the WLP if and only if $d > 1$.*

This conjecture has been solved for $d \neq 3$ and $n = 3$ in [37] and for all other cases in two steps by work of Nagel and Trok [41]; and Boij and Lundqvist [14]. So, we are let to pose the following problem:

Problem 2.12. *Let $R = \mathbb{K}[x_1, \dots, x_m]$. Let $L_1, \dots, L_r \in R$ be general linear forms, and let $I = \langle x_1^d, \dots, x_m^d, L_1^d, \dots, L_r \rangle$. Determine for which values of m, d and r the algebra R/I fails t.he WLP*

Gotzmann ideals. A square-free monomial ideal $I \subseteq S = \mathbb{K}[x_0, \cdots x_n]$ is called *Gotzmann ideal* if and only if

$$I = m_1(x_i \ : \ i \in J_1) + m_1 m_2(x_i \ : \ i \in J_2) + \cdots + m_1 m_2 \cdots m_s(x_i \ : \ i \in J_s)$$

for some square-free monomials $m_1, \dots, m_s$ and pairwise disjoint subsets $J_1, \dots, J_s$ of $\{0, 1, \dots, n\}$.

The above definition was introduced by Hoefel and Mermin in [25] Herzog and Hibi [27] showed that all Gotzmann ideals are componentwise linear, and Bigdeli and Faridi [11] established a connection between square-free Gotzmann and Stanley-Reisner ideals of *chordal complexes* – a large class of componentwise linear ideals which can be defined via simplicial collapses. Open questions that arise from this are the following:

Question 2.13. *Which square free Gotzmann ideals satisfy the WLP? Do (some) Artinian reductions of Stanley-Reisner ideals of chordal complexes satisfy the WLP?*

See [26] for some recent contributions to this question and related problems.

3. Geometric aspects related to Lefschetz properties

In the last decade the failure or presence of the WLP has been connected to a large number of geometric problems, that appear to be unrelated at first glance. For example, in [36], Mezzetti, Miró- Roig and Ottaviani proved that the failure of the WLP is related to the existence of projective varieties satisfying at least one Laplace equation of order greater than 2. Another connection with classical algebraic geometry has been discovered by Maeno and Watanabe, who proved that an Artinian Gorenstein algebra $A = \mathbb{K}[x_1, \dots, x_n]/\mathrm{Ann}(F)$ fails the SLP if one of the higher Hessians of F is identically zero [35]. This motivates the study of projective hypersurfaces with vanishing Hessian or higher Hessians, a classical problem that goes back to Gordan and Noether. In this section we will exhibit other geometric properties closely related to the failure or presence of the WLP. As in the previous section we divide this section into several subsections.

Reduced sets of points. It is known that not every Artinian ideal is an Artinian reduction of the ideal of a reduced set of points, and certainly it is not a *general* Artinian reduction of the ideal of a reduced set of points. An open question is the following:

Question 3.1. *Does a general Artinian reduction of a reduced, arithmetically Gorenstein set of points in $\mathbb{P}^n$ has the WLP (provided that the characteristic of the underlying field equals zero)?*

A positive answer to this question would imply the algebraic g-conjecture and yields a complete classification of the Hilbert functions of such sets of points. We want to emphasize that it is important to consider *general* Artinian reductions since examples of reduced arithmetically Gorenstein sets of points exist for which a *special* Artinian reduction fails the WLP. A similar but less ambitious question whose positive answer would be implied by Question 3.1 is the following:

Question 3.2. *Is the h-vector of every reduced, arithmetically Gorenstein set of points unimodal (provided that the characteristic of the underlying field equals zero)?*

Artinian reductions of arithmetically Cohen-Macaulay curves. A question which is of the same flavour and completely wide open is the following:

Question 3.3. *Do all irreducible arithmetically Cohen-Macaulay curves in $\mathbb{P}^4$ have an Artinian reduction that has the SLP?*

Unexpected curves and complex line arrangements. Unexpected sets of points relate directly to failures of the SLP. In particular, a finite set $Z \subseteq \mathbb{P}^n$ of r points $p_1, \ldots, p_r$ dual to hyperplanes $H_1, \ldots, H_r$ has an *unexpected hypersurface cone* of degree d with a general point of multiplicity d if and only if $\mathbb{C}[x_0, \ldots, x_n]/(L_1^d, \ldots, L_r^d)$ fails SLP in degree $d-1$ with range 1 (see [23, Proposition 2.17]).

Unexpectedness in the plane often comes from line arrangements. In particular, there are four known kinds of arrangements of distinct lines $L_1, \ldots, L_r$ in $\mathbb{P}^2$ (over the complex numbers) such that whenever L_i and L_j meet there is a third line L_k meeting at the same point. They are:

- 3 or more concurrent lines,
- the Fermat arrangements $((x_0^r - x_1^r)(x_0^r - x_2^r)(x_1^r - x_2^r))$,
- the Klein arrangement of 21 lines, (this has 21 points where exactly 4 lines meet, 28 points where exactly 3 lines meet, and no other points where 2 or more lines meet),
- the Wiman arrangement of 45 lines (this has 36 points where exactly 5 lines meet, 45 points where exactly 4 lines meet, 120 points where exactly 3 lines meet and no other points where 2 or more lines meet).

For $r \geq 5$, the points dual to the lines of the Fermat arrangements have unexpected curves. The points dual to the Klein and Wiman lines also have unexpected curves (see [18]). This naturally raises the following open problem:

Question 3.4. *Are there other complex line arrangements having no points where exactly two lines meet?*

Unexpected hypersurfaces and companion varieties. The existence of an unexpected curve was first established for the points of the B_3 root system. In [43] it was shown that the unexpectedness is directly related to the existence of Togliatti-type surfaces (having defective osculating spaces). Moreover, the author showed, in the case of the surface associated to the B_3 root system, that there exists another surface, called a *companion surface*, which exhibits a number of interesting geometrical properties. This direction of study has been pursued in [20] in case of Togliatti-type varieties and their companions associated to the B_4, F_4 and H_3 root systems and to certain Fermat-type configurations of points. This motivates the following problem:

Problem 3.5. *Study Togliatti-type varieties and their companion surfaces for other root systems and find a Togliatti-type construction associated to configurations of points allowing multiple general fat points as described in* [44].

Sets of points projecting to complete intersections A finite set $Z \subseteq \mathbb{P}^n$ is said to be *geproci* if its image Z_P under projection from a general point P to a fixed hyperplane H is a complete intersection. An example is a set of points Z contained in a hyperplane L which is already a complete intersection in L. The only other examples of geproci sets known are in $\mathbb{P}^3$. In this case, we say Z is (a,b)-*geproci* if Z_p is the transverse intersection of a curve $C_{a,P}$ of degree a with a curve $C_{b,P}$ of degree b where $a \leq b$ (hence $|Z| = ab$). Non-degenerate examples of such sets are given by (a,b)-*grids*. A set Z of ab points in $\mathbb{P}^3$ is called an (a,b)-*grid* if there is a set A of a skew lines and a set B of $b \geq a$ skew lines, such that each line in A meets each line in B in exactly one point (see [16]). By a theorem of [15], if Z is an (a,b)-grid with $3 \leq a \leq b$, then $C_{a,P}$ and $C_{b,P}$ are unexpected curves and thus give failures of SLP (see the previous paragraph).

We define two more types of geproci sets. A *halfgrid* is a geproci set Z such that either $C_{a,P}$ or $C_{b,P}$ (but not both) is a union of lines for a general P. A *non-halfgrid* is a geproci set Z such that neither $C_{a,P}$ nor $C_{b,P}$ is a union of lines for a general P.

Since degenerate geproci sets and grids are easy to construct and well-understood, they are referred to as *trivial* in the folowling. The obvious question is whether other geproci sets exist. More precisely, the following problem is open:

Problem 3.6. *Find a non-trivial* (a,b)*-geproci set, where* $C_{a,P}$ *and* $C_{b,P}$ *is not unexpected, or show that no such example exists.*

So far, not a single example of a non-trivial (a,b)-geproci set where $C_{a,P}$ and $C_{b,P}$ are not unexpected, is known. Moreover, since every nontrivial geproci set known has at least one subset of 3 collinear points, the following problem is natural:

Problem 3.7. *Prove or disprove (by giving a counterexample) that every non-trivial* (a,b)*-geproci set has a subset of* 3 *collinear points.*

As a more general version, one can even consider the following problem:

Problem 3.8. *Find an example of a linearly general geproci set, or show that no such example exists.*

In fact, all non-trivial geproci sets known have a rich struccture of linearly dependent subsets. Associating a matroid to these subsets, gives rise to Terao type problems, for instance:

Question 3.9. *Are geproci sets with isomorphic matroids projectively equivalent? Is* being geproci *a property of the associated matroids, i.e., is a set of points whose matroid is isomorphic to the matroid of a geproci set, geproci itself?*

4. Lefschetz properties in combinatorics

Lefschetz properties have very nice applications to ombinatorics. The main goal of this section is to gather several open problems concerning Lefschetz problems for simplicial complexes and more precisely for their Stanley–Reisner ideals. As for the other sections, we divide this section into several topic areas.

We first fix some standard notation. Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the standard graded polynomial ring over a field $\mathbb{K}$ of characteristic zero. Usually Δ will denote a simplicial complex on vertex set $[n] = \{1, \dots, n\}$ and I_Δ, respectively $\mathbb{K}[\Delta]$ will denote the corresponding Stanley–Reisner ideal respectively Stanley-Reisner ring. These, themselves, are not Artinian rings, but knowing Lefschetz properties of Artinian reductions of Stanley–Reisner rings is an important problem from both algebraic and combinatorial viewpoints. One of the biggest progress on this topic is the following recent result of Adiprasito [4] (see also [6, 34, 42])

Theorem 4.1. *If Δ is a simplicial $(d-1)$-sphere, then for a generic linear system of parameters Θ of S/I_Δ, the Artinian algebra $S/(I_\Delta + (\Theta))$ has the SLP.*

The special case of this theorem that Δ is the boundary complex of a simplicial polytope is a classical result due to Stanley and follows from the Hard Lefschetz theorem for toric varieties (see [46, III §1]). It should also be mentioned that the theorem has a generalization to simplicial manifolds and doubly Cohen–Macaulay simplicial complexes (see [6]). While these works give answers to many unsolved problems, still a variety of open problems on this topic remains.

Balanced spheres A simplicial complex on $[n]$ of dimension $(d-1)$ is said to be *(completely) balanced* if there exists a function $c : [n] \to [d]$ such that $c(v) \neq c(u)$ for all edges $\{u, v\} \in \Delta$. It should be noted that the function c can be regarded as a proper *coloring* of the 1-skeleton, i.e., the graph of Δ. If Δ is balanced, then it is known that the sequence of linear forms $\theta_1, \dots, \theta_d$ defined by $\theta_i = \sum_{c(v)=i} x_v$ forms a linear system of parameters of Δ, called a *colored linear system of parameters* of Δ (see [46, II §4]). The following question is asked in [19, Conjecture 1.3].

Problem 4.2. *Let Δ be a balanced simplicial $(d-1)$-sphere and $\Theta = \theta_1, \dots, \theta_d$ be a colored linear system of parameters of Δ. Does the algebra $S/(I_\Delta + (\Theta))$ have the SLP (or WLP)?*

Centrally symmetric spheres A simplicial complex Δ on vertex set $[n]$ is said to be *centrally symmetric* (*cs*, for short) if it admits a free $\mathbb{Z}/2\mathbb{Z}$-action, that is, there is a fixed-point free involution

$\alpha : [n] \to [n]$ with $\Delta = \{\alpha(F) \mid F \in \Delta\}$. For such a simplicial complex, one can choose a linear system of parameters $\Theta = \theta_1, \dots, \theta_d$ from $S^- = \{f \in S \mid \alpha(f) = -f\}$, where $\alpha(x_v)$ is defined to be $x_{\alpha(v)}$.

Problem 4.3. *Let Δ be a centrally symmetric simplicial sphere and let $\Theta = \theta_1, \dots, \theta_d$ be a linear system of parameters for $\mathbb{K}[\Delta]$ that has been generically taken from S^-. Does the algebra $S/(I_\Delta + (\Theta))$ have the SLP or WLP with respect to the linear form $w = x_1 + \cdots + x_n$?*

The answer is known to be yes for cs simplicial polytopes (see [46, III §8] for more background).

Flag spheres A simplicial complex Δ is said to be *flag* if I_Δ has no generators of degree ≥ 3. h-vectors of flag simplicial spheres have been of great interest in combinatorics and there are two famous conjectures on this topic, known as *Charney-Davis Conjecture* and *Gal's gamma non-negativity Conjecture*. Motivated by these conjectures Hailun Zheng suggested the following question.

Problem 4.4. *Let Δ be a flag simplicial $(d-1)$-sphere with $d \geq 5$, let $\Theta = \theta_1, \dots, \theta_d$ be a generic linear system of parameters and let w, w' be additional generic linear forms. Is it true that the multiplication $\times w' : (S/(I_\Delta + (\Theta, w))_1 \to (S/(I_\Delta + (\Theta, w))_2$ is injective?*

Almost Lefschetz properties A special instance of a flag simplicial complex is the barycentric subdivisions $\mathrm{sd}(\Gamma)$ of a CW complex Γ. If Γ is a $(d-1)$-dimensional Cohen–Macaulay complex, then the following almost Lefschetz property is known (see [40, Theorem 6.2])

(i) $\times w^{d-2k-1} : \big(S/(I_{\mathrm{sd}(\Gamma)} + (\Theta))\big)_k \to \big(S/(I_{\mathrm{sd}(\Gamma)} + (\Theta))\big)_{d-1-k}$ is injective for $k \leq (d-1)/2$;
(ii) $\times w^{d-2k-1} : \big(S/(I_{\mathrm{sd}(\Gamma)} + (\Theta))\big)_{k+1} \to \big(S/(I_{\mathrm{sd}(\Gamma)} + (\Theta))\big)_{d-k}$ is surjective for $k \leq (d-1)/2$.

Here, Θ is a generic linear system of parameters, and w is a generic linear form. This result in particular implies that $S/(I_{\mathrm{sd}(\Gamma)} + (\Theta))$ has the WLP if d is odd.

Question 4.5. *Is there a Cohen–Macaulay CW complex Γ such that $S/(I_{\mathrm{sd}(\Gamma)} + (\Theta))$ does not have the weak Lefschetz property for any linear system of parameters Θ?*

As a related vague question, one could also ask

Question 4.6. *Are there other classes of simplicial complexes (different from barycentric subdivisions) that satisfy (i) and (ii) above?*

While property (i) is known to hold for more general subdivisions [5, Theorem 46], property (ii) seems a bit mysterious at this moment.

Nagata idealizations. The idealization construction by Nagata has been used in many cases to construct Artinian Gorenstein algebras failing the WLP, for example the famous example by R. Stanley. In terms of Macaulay inverse systems, we can construct the idealization of the canonical module of an Artinian level algebra with inverse system $\langle F_1, F_2, \dots, F_s\rangle$ as the Gorenstein algebra with inverse system $\langle \sum x_i F_i \rangle$ where $x_1, x_2, \dots, x_s$ are new variables. Here the forms $F_1, \dots, F_s$

are assumed to be homogeneous of degree d in variables $y_1, y_2, \ldots, y_n$. Preliminary studies of the Gorenstein algebra given by $\langle \sum x_i^e F_i \rangle$ show that for $e \geq d$ it satisfies the WLP for any forms F_i while for $e < d$ we have examples failing the WLP and examples satisfying the WLP. THerefore we are in front of an interesting problem:

Problem 4.7. *To determine whether the Gorenstein algebra given by $\sum x_i^e F_i$ has the WLP and/or the SLP.*

As a first contribution to this last problem the reader can look at [3] and [22].

5. Lefschetz properties and Jordan types of Artinian algebras

Let $\mathbb{K}$ be any field, and let A be an Artinian algebra, quotient of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ or of the local regular ring $\mathbb{K}\{x_1, \ldots, x_n\}$. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal ideal of A. Let M be a finite module over A, and let $\ell \in \mathfrak{m}$. The *Jordan type* $P_{\ell,M}$ of ℓ is the partition of $\dim_{\mathbb{K}} M$ giving the sizes of the Jordan blocks in a Jordan canonical form for the linear map $\times \ell : M \to M$, defined by the multiplication by ℓ. There are several refinements of this invariant. We refer to [29] in this volume for their definitions and a discussion of their properties. In particular, when M is a graded module over an Artinian algebra A we define the generic linear Jordan type $P_{1,M}$ as the Jordan type $P_{\ell,M}$ for a generic linear form $\ell \in A_1$.

The Jordan type has two main strengths. First, in the graded case, it is a finer invariant than the Lefschetz properties: we can determine if a linear form ℓ satisfies the WLP or the SLP on A from its Jordan type and the Hilbert function $H = H(A)$: that is, ℓ has the SLP if its Jordan type is the conjugate $H^\vee$; and ℓ has the WLP if the number of parts of the Jordan type is the maximum value of H. Second, the Jordan type is well-defined also on non-graded algebras, allowing us to generalize the definitions of WLP and SLP to the non-graded case. As in the previous sections we distinguish several topics:

Linear Jordan type and contiguous Jordan type. Let A be standard graded, if $H = H(A)$ is unimodal (and has a single maximum value with no dips) and if the generic Jordan type is the conjugate partition of H (i.e., A is strong Lefschetz) then the generic linear Jordan type is equal to $H^\vee$, and therefore they are the same [30, Proposition 2.14].

Question 5.1. *Is this equality still true when $H(A)$ is not unimodal, or when the generic Jordan type is not the conjugate partition of the Hilbert function? More generally, under which conditions on a graded module M over a graded Artinian algebra A do their generic and generic linear Jordan types satisfy $P_M = P_{1,M}$? (See Questions 1.1 and 2.56 in* [30]*.)*

We can decompose M as the direct sum of $\mathbb{K}[\ell]$ modules – called ℓ-strings – whose lengths are given by the Jordan partition $P_{\ell,M}$, The Jordan degree type JDT $P_{\deg,\ell,M}$ of a graded module M adds to the Jordan type the information about the initial degree of the ℓ-strings in such a decomposition of M as module over $\mathsf{k}[\ell]$, this depends only on the pair (ℓ, M). However, a problem is that this definition of JDT does not generalize to non-graded Artinian algebras or modules [31].

When the Hilbert function H of a graded Artin module is non-unimodal, we can define a contiguous-Jordan type $P_{c,\ell}(H)$, and a contiguous-Jordan degree type $P_{c,\deg,\ell}(H)$: using the bar graph of $H(M)$ [30, Definition 2.28].

Question 5.2. *For which (non-unimodal) Hilbert functions H occurring for graded Artinian algebras A, can we find pairs (A, ℓ) with $H(A) = H$ and the Jordan type and Jordan degree types of A agree with the contiguous Jordan or Jordan-degree type of H?*

Jordan type of the initial ideal. Recall that the *initial ideal* $\operatorname{init}(I)$ of an ideal $I \subseteq R = \mathbb{K}[x_1, \dots, x_n]$ is the ideal generated by the leading terms of elements of I under a fixed monomial order. From a result in [17] and [48] we know that if $R/(\operatorname{in}(I))$ has the WLP (resp. the SLP), then the same holds for R/I. More precisely, for a generic linear form ℓ we have that for all j, k the following inequality is true

$$\dim_{\mathbb{K}}(R/(I, \ell^k)_j \leq \dim_{\mathbb{K}}(R/(\operatorname{in} I, \ell^k))_j.$$

Question 5.3. *How does the generic Jordan degree type of a standard graded Artinian algebra $A = R/I$ behave under projection to the quotient $R/\operatorname{in}(I)$?*

Is there a pair (A, ℓ), where A is a standard graded Artinian algebra and $\ell \in A_1$, such that the Jordan degree type $P_{\deg,\ell,A}$ cannot occur for a pair (A', ℓ'), if A' is defined by a monomial ideal?

Jordan type for non-graded algebras. Lately, some refinements of Jordan type have been introduced for non-graded Artinian algebras: namely sequential Jordan type, Loewy sequential Jordan type, or double sequential Jordan type [31]. Each of these is semicontinuous in a family of algebras having fixed Hilbert function. The semicontinuity of Jordan type has been used to show that certain families $\operatorname{Gor}(H)$ of Artinian Gorenstein algebras with given Hilbert function have several irreducible components, beginning in codimension three [28].

Question 5.4. *Determine Hilbert function sequences H for non-graded Artinian algebras such that the family $Z(H)$ of all algebras with Hilbert function H has several irreducible components Ξ_1, Ξ_2-whose general elements have the same generic Jordan type, but which differ in one of the refined invariants?*

Question 5.5. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of finite modules over A. How can we compare the generic Jordan types P_L, P_M, and P_N? Under what conditions could we have additivity $P_M = P_L + P_N$, in a suitable sense for the Jordan types? The same question can be extended to Jordan degree type, in the graded case, or to sequential Jordan type, Loewy sequential Jordan type, or double sequential Jordan type, in the non graded case.*

The next question has to do with the symmetric decomposition of the associated graded algebra of a non-graded Gorenstein Artinian algebra (see [28]). We know, by an example of Chris McDaniel [31] that it is not always possible to find a Jordan basis B for the multiplication by an element

$\ell \in \mathfrak{m}$ that agrees with the Hilbert function of A, in the sense that $B \cap \mathfrak{m}^i$ is a basis for the vector space $\mathfrak{m}^i$ for every i. We also know that there is always a pre Jordan basis that agrees with the Hilbert function in this sense [31]. So we are let to pose the following question:

Question 5.6. *If the Artinian algebra A is Gorenstein, can we always find a pre Jordan basis B that agrees with the $Q(a)$ decomposition, in the sense that $B \cap \mathfrak{m}^i \cap (0 : \mathfrak{m}^b)$ is a basis for the vector space $\mathfrak{m}^i \cap (0 : \mathfrak{m}^b)$ for every i and b?*

Jordan type of graded Artinian Gorenstein algebras. Graded Artinian Gorenstein (AG) algebras satisfy Poincaré pairing. This allows us to determine their Lefschetz properties by the ranks of fewer multiplication maps. For instance, a graded AG algebra A satisfies the WLP if the multiplication map by a generic linear form has maximal rank in degree $\lfloor \frac{d}{2} \rfloor$, where d is the socle degree of A (see [39] where it is also shown that WLP can be sensitive to characteristic). In codimension three, when the characteristic of $\mathbb{K}$ is zero, each standard-graded complete intersection algebra has the WLP [24]; some have conjectured that all codimension three standard graded AG algebras have the WLP (see [12] where it is proven that this conjecture depends on showing the special cases of compressed algebras). It is open which graded AG algebras of codimension three have the SLP [1]. There are families of graded AG algebras of codimension four that fail the WLP [2, Theorem 1.6]. There are AG algebras in codimension five whose Hilbert functions are non-unimodal, and thus are also not WLP. The question of whether there are complete intersection algebras in codimension four (or even higher) that fail WLP has been asked by many researchers and is open.

Question 5.7. *Determine the number of parts of generic linear Jordan types for the families of graded Artinian Gorenstein algebras failing the WLP. More generally, determine the generic linear Jordan type for a graded Artinian Gorenstein algebra that fails the SLP.*

Question 5.8. *Given a sequence H that can occur as the Hilbert function of a graded Artinian Gorenstein algebra, can we find potential Jordan types or Jordan degree types? (Known in codimenson two* [10]*).*

Question 5.9. *In* [10]*, all possible linear Jordan types of complete intersection algebras of codimension two having a fixed Hilbert function is listed. For every such Hilbert function H, can we determine the pairs of Jordan types P and Q that occur for (ℓ_1, A) and (ℓ_2, A) where $H(A) = H$?*

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