JORDAN TYPE OF AN ARTINIAN ALGEBRA
PROBLEM SESSIONS

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1. Preliminaries

Exercise 1.1. Consider the Artinian algebra $A = \mathsf{k}[x,y]/(x^2, xy^2, y^5)$, let $\mathsf{m} = (x,y)$ be its maximal ideal.

(1) Find the Jordan types of y^2, xy, and $xy + y^3$ in A.
(2) Write down a Jordan basis for y^2, and the corresponding Jordan canonical form.
(3) Describe the locus of the Jordan type $(5,2)$, i.e. the set
$$L_{(5,2)} = \{\ell \in \mathsf{m} : P_{\ell,A} = (5,2)\}.$$
(4) Suppose k is infinite. Which is the generic Jordan type of A?

Exercise 1.2. Consider the Artinian algebra $A = \mathsf{k}[x,y,z]/(yz - x^3,\ y^3,\ z^2)$, a non--graded complete intersection.

(1) Suppose $\operatorname{char}\mathsf{k} = 2$ and compute the Jordan type of $\ell_1 = x + y + z$.
(2) Suppose $\operatorname{char}\mathsf{k} \notin \{2,3,5\}$ and write $\ell = ax + by + cz + \ell'$, with $\ell' \in \mathsf{m}^2$. Show that $P_{\ell,A} = (7,5,3,3)$ if and only if $ab \neq 0$.

Exercise 1.3. Let A be an Artinian local algebra, with socle degree j, let m be its maximal ideal, and suppose that $\mathsf{k} = A/\mathsf{m}$ is an infinite field, with $\operatorname{char}\mathsf{k} = 0$ or $\operatorname{char}\mathsf{k} > j$.

(1) Show that there is an element $\ell \in \mathsf{m}$ such that $\ell^j \neq 0$.
(2) Show that if $\ell' \in \mathsf{m}^2$ then its Jordan type is not the generic Jordan type of A.

Exercise 1.4. Consider the algebra $A = \mathsf{k}[x,y]/(x^4, y^3)$. Use the Lemma 2.11 to compute the Jordan type of x.

Exercise 1.5. Consider the algebra $A = \mathsf{k}[x,y,z]/(xy,\ xz,\ yz,\ x^j - y^j,\ x^j - z^j)$. Compute the Jordan type of x.

Exercise 1.6. Let $A = \mathsf{k}[x,y]/I$ be an Artinian algebra, and let $P_{x,A} = (p_1, \dots, p_s)$. Consider the algebra $B = \mathsf{k}[x,y,z]/J$, where $J = (xz, yz, z^j, I)$. What is the Jordan type $P_{x,B}$?

2. Lefschetz properties

Exercise 2.1. Consider the Artinian algebra $A = \mathsf{k}[x,y]/(x^2, y^2)$.

(1) List all partitions of 4 that are dominated by $H(A)^\vee$.
(2) Confirm that all partitions are attained as the Jordan type of an element in the maximal ideal.

Exercise 2.2. Consider the Artinian algebra $A = \mathsf{k}[x,y]/(x^2, xy^2, y^5)$, let $\mathsf{m} = (x,y)$ be its maximal ideal. Give an example of a partition that is dominated by $H(A)^\vee$ but is not the Jordan type of any element $\ell \in \mathsf{m}$.

3. Artinian Gorenstein algebras

Exercise 3.1. Consider the Artinian Gorenstein algebra

$$A = \mathsf{k}\{x,y\}/(x^3 - y^5,\, x^2y^2).$$

Write down a monomial basis for A, where the order of each element is apparent (i.e if $m_1, m_2 \in \mathsf{k}\{x,y\}$ are two monomials such that $m_1 = m_2$ in A, choose the one with highest degree).

Exercise 3.2. Consider the Artinian Gorenstein algebra $A = \mathsf{k}\{x,y,z\}/\operatorname{Ann} F$, for $F = X^2Y^2 + Z^3$.

(1) Compute its $Q(a)$ modules, and get the Hilbert function of A as the sum of its symmetric decomposition.
(2) Check that $\operatorname{Ann} F = (xz,\, yz,\, x^3,\, y^3,\, z^3 - x^2y^2)$.
(3) Compute the Jordan type $P_{x+y+z,A}$. Is this a weak or strong Lefschetz element?
(4) Find two Jordan types of elements of order one (i.e. elements in $\mathsf{m} \setminus \mathsf{m}^2$) that are not comparable.

Exercise 3.3. Let A be an Artinian Gorenstein algebra, with maximal ideal m.

(1) Let $z \in \mathsf{m}^k \cap (0 : \mathsf{m}^c)$, and $\ell \in \mathsf{m}^i$, with $c > i$. Show that
$$\ell z \in \mathsf{m}^{i+k} \cap (0 : \mathsf{m}^{c-i}).$$
(2) Suppose z represents a non-zero class in $Q(a)_k$ and $\ell \in \mathsf{m}$. Show that if ℓz represents a non-zero class in $Q(b)_m$, and $b < a$ then $m \geq k+2$.

Exercise 3.4. Let A be an Artinian Gorenstein algebra with Hilbert function

$$H(A) = (1,3,3,4,2,1).$$

(1) Compute the Hilbert functions of its $Q(a)$ modules.
(2) Use Exercise 3.3 to show that A is not strong Lefschetz.

4. Finer invariants, Jordan type, and their behaviour under deformations

Exercise 4.1. Let $R = \mathsf{k}[x,y,z]$, and let

$$A = R/(y^2, x^2z, x^2y, z^3, x^6), \text{ and } B = R/(yz, x^2y, xy^2, z^3, x^4, y^4).$$

(1) Show that $P_{z,A} = P_{z,B}$.
(2) Compute the Jordan degree types $\mathcal{S}_{z,A}$ and $\mathcal{S}_{z,A}$.

Exercise 4.2. Consider the Artinian Gorenstein algebra $A = \mathsf{k}\{x,y,z\}/\operatorname{Ann} F$, for $F = X^4 + XY^3 + ZY^3$. Use the rank matrix to compute the Jodan degree type of x.

Exercise 4.3. Consider the Artinian Gorenstein algebra $A = \mathsf{k}\{x,y,z\}/\operatorname{Ann} F$, for $F = X^4Y^2 + ZY^3$. Compute the sequential Jodan degree type of y.