LEFSCHETZ PROPERTIES IN ALGEBRA, GEOMETRY AND COMBINATORICS

EXERCISES

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Exercise 1 Prove that in the ring $R = k[x_1, \dots, x_n]$ there are $\binom{d+n-1}{n-1}$ monomials of degree d for any $d \geq 1$ and $n \geq 1$.

$h_{R/I}(t) = \dim R_t - \dim I_t$

$k[x, y, z]_6 \ni x^3 y z^2 \longleftrightarrow (3, 1, 2)$

4

$\bullet \bullet \bullet \mid \bullet \mid \bullet \bullet$

$6 \bullet + 2 \mid = 8$

$\binom{8}{6} = \binom{8}{2}$

$h_R(d) = \binom{d+n-1}{n-1}$

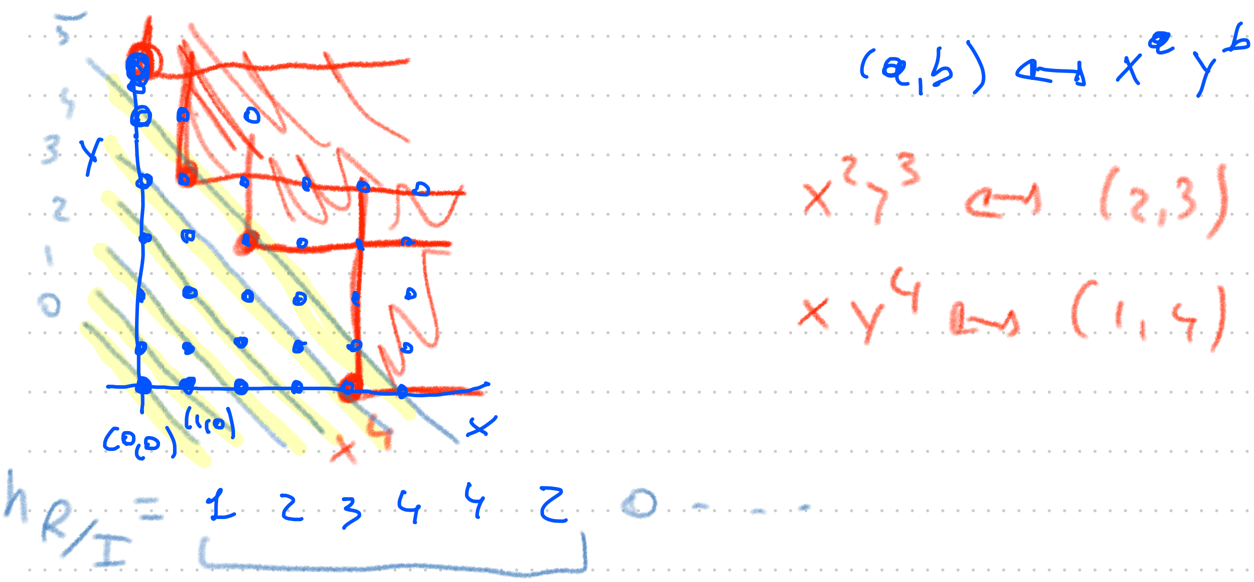
Exercise 36 Let $R = k[x, y]$ and let $I = \langle x^4, x^2y^3, xy^4, y^6 \rangle$.

(a) Draw a picture, using the integer points in the first quadrant and shading, to represent the monomials in I.

(b) What are the monomials *not* in I?

(c) What is the Hilbert function of R/I?

(d) What is the Hilbert polynomial of R/I?



Exercise 37 Is the following an O-sequence?

$$(1, 5, 12, 17, 25, 36)$$

Exercise 24 Let ϕ be an automorphism of $\mathbb{P}^2$. What this means is that there is some *invertible* 3×3 matrix A such that for $P = [p_1, p_2, p_3]$,

$$\phi(P) = A \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

Let P, Q, R be three points in $\mathbb{P}^2$. Show that if P, Q, R are collinear then $\phi(P), \phi(Q), \phi(R)$ are collinear. Is the converse true?

$$ax + by + cz = 0$$

$$(a, b, c) \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = 0$$

$$A \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \varphi(P)$$

$$(a, b, c)\, I \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

↓

$$(a, b, c) \cdot A^{-1} \cdot A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

1×3 $\quad 3 \times 3$

1×3

$$(a', b', c') \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Exercise 49 An important tool for studying Lefschetz properties for *monomial* algebras is the fact that R/I has the WLP (or SLP) if and only if the linear form given by the sum of the variables is a Lefschetz element.

R/I has WLP

$a_1x_1 + a_2x_2 + \cdots + a_nx_n \in R_1 \qquad a_1, a_2, \ldots, a_n \neq 0$

$\longrightarrow x_1 + x_2 + \cdots + x_n$

I

$x_1 \mapsto \frac{x_1}{a_1}$

$x_2 \mapsto \frac{x_2}{a_2}$

$\cdots$

$x_n \mapsto \frac{x_n}{a_n}$

$x_1^{e_1} \cdots x_n^{e_n} \mapsto \frac{x_1^{e_1}}{a_1^{e_1}} \cdots \frac{x_n^{e_n}}{a_n^{e_n}} \in I$

$K[x, y, z] \qquad ax + by + cz \qquad a \neq 0,\ b \neq 0,\ c \neq 0$

$ax + by = 0 \qquad ax + cz = 0 \qquad by + cz = 0$

Exercise 26 A beautiful fact about projective space is the notion of **duality**. Let's limit ourselves to the real projective plane $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{R}}$

Recall that a line ℓ in $\mathbb{P}^2$ is the vanishing locus of a homogeneous linear polynomial, i.e. $\ell = \mathbb{V}(ax + by + cz)$ for some choice of $a, b, c \in \mathbb{R}$ not all zero.

(a) Show that $ax + by + cz = 0$ defines the same line as $3x + 4y + 5z = 0$ if and only if there exists some $t \in \mathbb{R}$ such that $a = 3t$, $b = 4t$ and $c = 5t$. (Of course 3, 4, 5 is just an example.) [Hint: $\Leftarrow$ is almost immediate. For $\Rightarrow$, you can use the fact that in $\mathbb{P}^2$, either two lines meet at a single point or they are the same line. It may help to take the linear algebra point of view.]

(b) Based on (a), show that the **set** of lines in $\mathbb{P}^2$ itself can be viewed as a projective plane, which we will denote by $(\mathbb{P}^2)^\vee$.

$ax + by + cz \longleftrightarrow (a, b, c) \in \mathbb{P}^2$

$\mathbb{P}^2 = (\mathbb{P}^2)^{\vee\vee} \qquad (\mathbb{P}^2)^\vee$

$(a, b, c) \longrightarrow ax + by + cz$

$\longleftarrow \quad (a, b, c)$

(c) Let P_1, P_2, P_3 be points of $(\mathbb{P}^2)^\vee$ and let $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ be the lines in $\mathbb{P}^2$ that they correspond to. Show that P_1, P_2, P_3 all lie on a line in $(\mathbb{P}^2)^\vee$ if and only if $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ all pass through a common point.

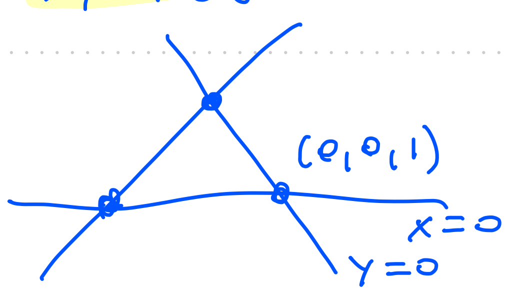
(d) Using (c), if you take a **line** in $(\mathbb{P}^2)^\vee$, what does the collection of all the points on this line correspond to back in $\mathbb{P}^2$? Explain your answer carefully.

$\mathbb{P}^2 \qquad (\mathbb{P}^2)^\vee$

$ax + by = 0 \longrightarrow (a, b, 0) \quad z = 0$

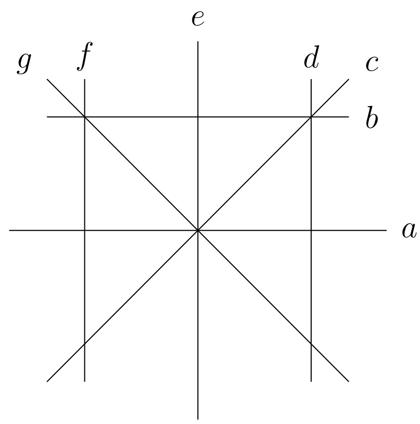
$ax + cz = 0 \longrightarrow (a, 0, c) \quad y = 0$

$by + cz = 0 \longrightarrow (0, b, c) \quad x = 0$



$\longrightarrow \quad z = 0$

(e) The following is a set of lines in $\mathbb{P}^2$, labelled a to g.



Sketch the set of points in $(\mathbb{P}^2)^\vee$ dual to these lines, and label them A to G **corresponding to the similarly named lines**. Make sure that your sketch reflects when three or more of the points are on a line. [Hint: in addition to the obvious places where three or more lines meet, the three vertical lines meet at infinity!! Part (c) is crucial in this problem.]

Exercise 50 Let $I = \langle x^2, y^2, z^2 \rangle \subset R = k[x, y, z]$.

R/I

(a) Prove that the Hilbert function of R/I is $(1, 3, 3, 1)$ (writing only the non-zero values).

0	1	2	3	4
1	x	xy	xyz	0
	y	xz		
	z	yz		

$[R/I]_1 \quad [R/I]_2$

(b) Let $L = x + y + z$. Show that $\times L$ is injective from degree 0 to degree 1 and surjective from degree 2 to degree 3.

$1 \overset{\times L}{\longmapsto} x + y + z \in R_1$

$xy \overset{\times L}{\longmapsto} x^2 y + x y^2 + xyz$

(c) Show that $\times L$ is bijective from degree 1 to degree 2 if and only if $\mathrm{char}(k) \neq 2$. Combining (b) and (c), conclude that R/I has the WLP if and only if $\mathrm{char}(k) \neq 2$.

$x \mapsto x^2 + xy + xz$

$y \mapsto xy + y^2 + yz$

$z \mapsto xz + yz + z^2$

$\times L : (R/I)_1 \to (R/I)_2$

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -1 - 1 = -2 \neq 0$$

if $\mathrm{ch}(k) \neq 2$

if $\mathrm{ch}(k) = 2$

$= 0$

(d) If $\text{char}(k) = 2$, find an element in $[R/I]_1$ which is in the kernel of $\times(x + y + z)$ from degree 1 to degree 2.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise 51 [Brenner-Kaid] Let $R = k[x, y, z]$ and $I = \langle x^3, y^3, z^3, xyz\rangle$.

(a) Prove that R/I is artinian.

(b) Find the Hilbert function of R/I.

(c) Show that R/I fails the WLP in any characteristic. (Hint: focus on the multiplication from degree 2 to degree 3. It is a fact, which you can use, that for studying WLP for a monomial ideal, it is enough to consider $\times L$ for $L = x + y + z$.)