Exercise 28 Find the saturation of each of the following ideals (or explain why it is already saturated).

(a) $\langle x^2, y^2, z^2 \rangle \subset k[x, y, z]$.
(b) $\langle x^2, y^2, z^2 \rangle \subset k[w, x, y, z]$.
(c) $\langle x^2, xy, xz \rangle \subset k[x, y, z]$.

$I^{SAT} = (f \in R \mid x_i^{a_i} \cdot f \in I \ \forall i \ \exists a_i > 0)$

(a) $I = (x^2, y^2, z^2) \subseteq k[x,y,z] = R$

$1 \quad x, y, z \quad xy, xz, yz \quad xyz$

$xyz \in I^{SAT} \qquad I_4 = R_4$

$1 \cdot x^2 \in I$
$1 \cdot y^2 \in I$
$1 \cdot z^2 \in I \qquad 1 \in I^{SAT} = R$

(b) $I = (x^2, y^2, z^2) \subseteq k[w, x, y, z]$

$I = I^{SAT}$

$m \notin I \qquad m w^d \in I \iff m \in I$

(c) $I = (x^2, xy, xz) \subseteq k[x, y, z]$

1   3 → 3   4   5 ...

$1 \quad x, y, z \quad y^2, yz, z^2 \quad y^3, y^2z, yz^2, z^3 \quad y^a z^b \ldots$

R/I

$x \cdot x \in I$
$x \cdot y \in I$
$x \cdot z \in I \implies x \in I^{SAT} = (x)$

$[I]_d = [I^{SAT}]_d \quad d \geq 2$

$\ell \cdot x \in I$

Example 3.5. Let $R = k[x, y, z]$ and $I = \langle x^2, xy, xz \rangle$. Then the Hilbert function of R/I begins with the sequence $(1, 3, 3, \dots)$ but clearly $x \in [R/I]_1$ is in the kernel of multiplication by any linear form L. Notice also that the vanishing locus of I is not zero-dimensional, as might have been suggested by the fact that the Hilbert function is equal in degrees 1 and 2, but instead consists of the line $x = 0$.

Exercise 30. Find the entire Hilbert function of the algebra given in Example 3.5. Is there any other degree in which $\times L$ fails to be injective? Find the saturation of this ideal. What subvariety of $\mathbb{P}^2$ corresponds to this saturation?

Exercise 29. Show that the ideal I_V is a saturated ideal.

$f \in I_V^{SAT}$

$f \cdot x_i^{a_i} \in I_V \quad \forall i \quad \exists a_i$

$f \cdot x_i^{a_i}(P) = 0 \qquad \forall P \in V$

$\quad \| $

$f(P) \cdot \underbrace{x_i^{a_i}(P)}_{\neq 0} \quad \Rightarrow \quad f(P) = 0$

$\Rightarrow f \in I_V$

Exercise 33. Let $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$, the irrelevant ideal in the graded ring $R = k[x_0, \dots, x_n]$. Let I be a homogeneous ideal. Define

$$I : \mathfrak{m} = \{ f \in R \mid fm \in I \text{ for all } m \in \mathfrak{m} \}.$$

(a) Verify that $I : \mathfrak{m}$ is a homogeneous ideal in R.

(b) Show that I is saturated if and only if $I : \mathfrak{m} = I$.

$I \subseteq I : \mathfrak{m}$

$I = I^{SAT}$

($\Rightarrow$) $f \in I : \mathfrak{m} \to x_i \cdot f \in I \ \forall i. \to f \in I^{SAT} = I$

($\Leftarrow$) $f \in I = I : \mathfrak{m}.$ $[I]_d = [I^{SAT}]_d \quad d \gg 0$

$I \neq I^{SAT}$ $[I]_t \neq [I^{SAT}]_t$ t The bigger s.t.

$x_i \cdot f \in I$ $f \in I \setminus I^{SAT}$

$f \in I : \mathfrak{m} = I$

(c) We define a *socle element* of R/I to be a non-zero element $f \in [R/I]_t$ (for some t) such that f is annihilated by $\mathfrak{m}$. This corresponds to an element $f \in [I : \mathfrak{m}]_t \backslash [I]_t$. In particular, f is in the kernel of $\times L : [R/I]_t \to [R/I]_{t+1}$ for all $L \in [R]_1$. Show that I is saturated if and only if R/I has no socle.

$I : \mathfrak{m}$ $\quad f \in R/I : \quad f \cdot \mathfrak{m} \subseteq I$

$f \in [R/I]_t \quad f \cdot \mathfrak{m} \subseteq I$

Exercise 34. Prove that if $\operatorname{depth}(R/I) \geq 1$ then I is saturated.

regular element

F non-zero divisor in R/I

ℓ non-zero divisor in R/I

$F \in I \neq I^{SAT}$

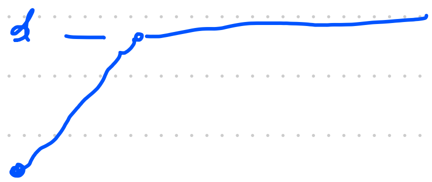
$F \in I \neq I : \mathfrak{m}$

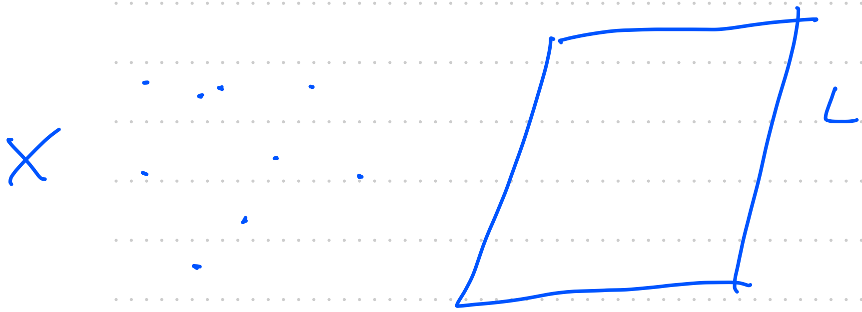
$F \cdot x_i \in I \quad \forall x_i$

$F \cdot x_i \to 0$

$F \cdot \ell$

Exercise 41. Prove that the Hilbert function of a set of d points in $\mathbb{P}^n$ is strictly increasing until it reaches the value d, at which time it becomes constant. Thus the Hilbert polynomial of a finite set of points is the constant polynomial equal to the number of points in the set.





L is a regular element in R/I_X

$$0 \to R/I_X(-1) \xrightarrow{\times L} R/I_X \to R/I_X+(L) \to 0$$

$$0 \to \left[R/I_X\right]_{t-1} \xrightarrow{\times L} \left[R/I_X\right]_t \to \left[R/I_X+(L)\right]_t \to 0$$

$$\dim \left[R/I_X\right]_{t-1} - \dim \left[R/I_X\right]_t + \dim \left[R/I_X+(L)\right]_t = 0$$

$$h_X(t-1) - h_X(t) + h_{R/I_X+(L)}(t) = 0$$

$$h_{R/I_X+(L)}(t) = h_X(t) - h_X(t-1) = \Delta h_X(t)$$

Δh_X is an O-sequence.

h_X is a O-sequence $\quad h_X = h_{R/I_X}$

$d-1 \quad h_X(d-1) = d$

$X = \{P_1, \ldots, P_d\} \subseteq \mathbb{P}^n$

$$\dim \left[R/I_X \right]_{d-1} = d$$

$$\varphi : \left[R/I_X \right]_{d-1} \longrightarrow \mathbb{K}^d$$

$$\varphi(\bar{F}) = \left(\frac{\bar{F}(P_1)}{L^{d-1}(P_1)}, \ldots, \frac{\bar{F}(P_d)}{L^{d-1}(P_d)} \right)$$

$L(P_i) \neq 0$

$\forall P_i \in X$

$$\varphi(\bar{F}) = (0, \ldots, 0)$$
$$\Downarrow$$
$$\bar{F}(P_1) = \cdots = \bar{F}(P_d) = 0$$

$P_1 \; \cdots \; P_d$

$L_1 \quad\quad L_d$

$$F \in I_X \Leftrightarrow \bar{F} = 0$$

$$L_i(P_j) = \begin{cases} 0 & i = j \\ \neq 0 & i \neq j \end{cases}$$

$$(a, 0, \ldots, 0)$$

$F = \underbrace{L_2 \cdot \ldots \cdot L_d}$ $\in R_{d-1}$

$\notin I_X$

$$\varphi(\bar{F}) = \left(\frac{\bar{F}(P_1)}{L^{d-1}(P_1)}, \frac{\bar{F}(P_2)}{L^{d-1}(P_2)}, \ldots, \frac{\bar{F}(P_d)}{L^{d-1}(P_d)} \right)$$

(with $\bar{F}(P_2)$ and $\bar{F}(P_d)$ marked $\downarrow 0$)

$$= (e, 0, \ldots, 0)$$

$e \neq 0$

X is a finite set of points in $\mathbb{P}^n$ $\qquad h_X(1) = n+1$

i) h_X is an O-sequence $\quad R/I_X$

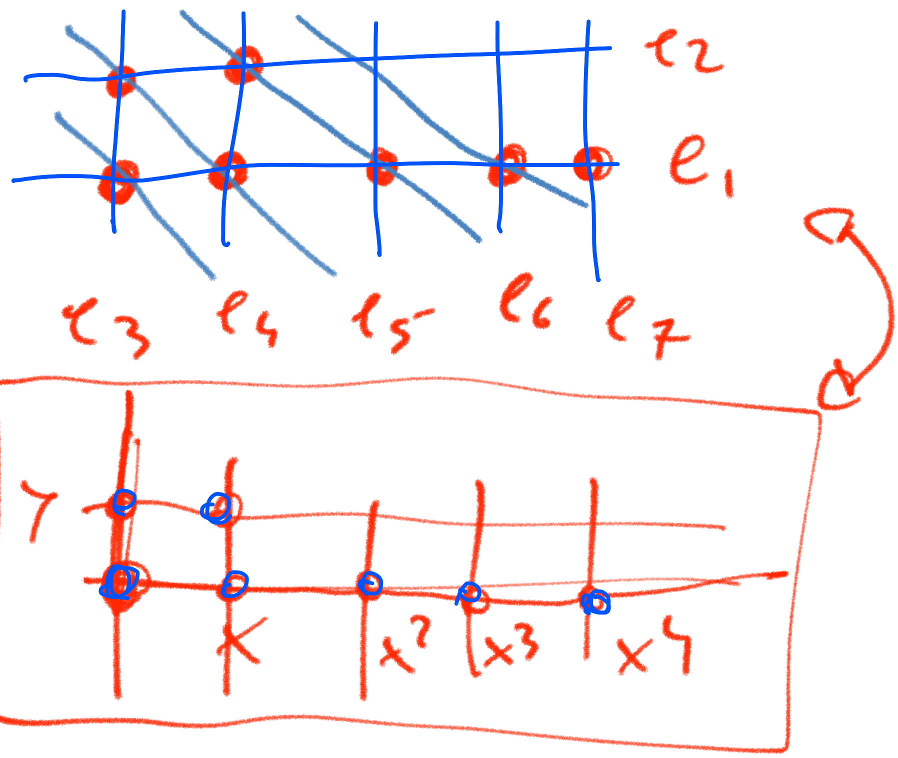
ii) Δh_X is an O-sequence. $\quad R/I_{X+(L)}$

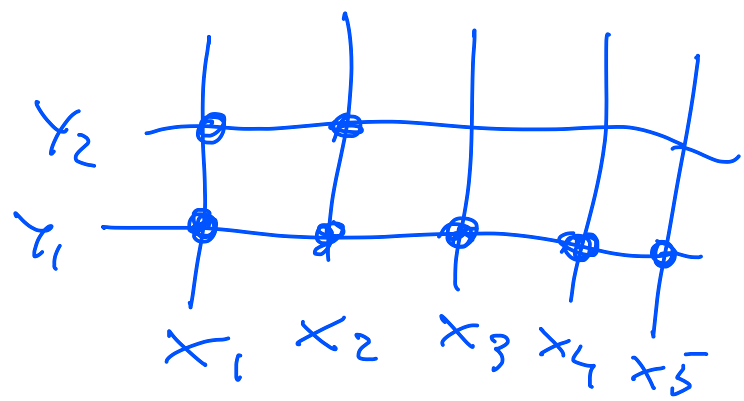
iii) $h_X(t) = |X| \quad t \gg 0$

$\Longleftrightarrow$ h is the H.F. of a finite set of points.

$h = \;1 \;\; \boxed{3} \;\; 5 \;\; 6 \;\; 7 \;\; 7 \; \cdots \; \mathbb{P}^2$

$\Delta h = \;1 \;\; 2 \;\; 2 \;\; 1 \;\; 1 \;\; 0$





$I = (x^5, x^2 y, y^2) \subseteq R[x, y]$

depth 0

$\tilde{I} = (x_1 x_2 x_3 x_4 x_5,\; x_1 x_2 y_1,\; y_1 y_2) \subseteq R[x_1, \dots, x_5, y_1, y_2]$

depth 5

$R[x_1, \dots, x_5, y_1, y_2, X, Y, Z]$

depth 8 $\quad K[X, Y, Z]$

$l_1 \ldots l_7$ general linear forms

$(x_1 - l_1 \ \ldots \ x_5 - l_5 \quad y_1 - l_6 \quad y_2 - l_7)$ $\quad$ in $X\,Y\,Z$

depth 1 $\quad K[\mathbb{P}^2]$

$1 \quad 3 \quad 3 \quad 3 \ldots$

$I = (x^2, xy, xz, y^3) \subseteq K[x, y, z] = R$

$h_{R/I}$

0	1	2	3	
1	3	3	3	…

1	x	y^2	y^2z	y^2z^2
	y	yz	yz^2	
	z	z^2	z^3	

Exercise 47. Let $R = k[w, x, y, z]$ and suppose $I \subset R$ is a homogeneous ideal with Hilbert function

$$h_{R/I}(t) = (1, 4, 3, 4, 5, \dots).$$

Prove that I is not saturated, and describe geometrically the saturation I^{sat} of I, and find its Hilbert function.

$\exists$ L reg. deg. R/I

$$0 \to R/I(-1) \xrightarrow{\times L} R/I \to R/I+(L) \to 0$$

$$h_{R/I+(L)} = \Delta h_{R/I}$$

$$\Delta h = 1 \quad 3 \quad -1 \quad 1 \quad 1 \quad \dots$$

$$[I^{sat}]_t = [I]_t \quad t \gg 0$$

$$h_{R/I^{sat}} = (1, 2, 3 \dots)$$