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Lecture 1

k = field (usually char 0)

$R = k[x_1, -, x_n]$

$= \bigoplus_{t \geq 0} [R]_t$ $\qquad$ $[R]_t$ = vector space of homog. poly. of degree t.

$\qquad\qquad\qquad\qquad$ = " " " forms of degree t

$[R]_m \cdot [R]_p \subseteq [R]_{m+p}$

R is a standard graded k-algebra

$$\dim [R]_t = \binom{n-1+t}{n-1}$$

Every ideal is finitely generated

homogeneous ideals - can choose the generators to be homogenous.

$$I = \bigoplus_{t \geq 0} [I]_t$$

R/I = quotient ring

$= \bigoplus_{t \geq 0} [R/I]_t$ $\qquad$ where $\qquad$ $[R/I]_t \cong [R]_t / [I]_t$

Def

The Hilbert function $h_{R/I}(t)$ is defined by

$$h_{R/I}(t) = \dim [R/I]_t$$
$$= \dim [R]_t - \dim [I]_t$$

Example / Def

Let $R = k[x_1, -, x_n]$. $\quad$ $I \subseteq R$ is a monomial ideal if there's a generating set consisting of monomials

A basis for $[R/I]_t$ can be obtained by listing the monomials of degree t not in I

Ex $I = (x^2, xy, xz, yz^2, y^3, z^4) \subset k[x,y,z]$

degree	Monomials in $[I]_t$	monomials not in $[I]_t$	$h_{R/I}(t)$
0	—	1	1
1	—	x, y, z	3
2	x^2, xy, xz	y^2, yz, z^2	3
3	...	y^2z, z^3	2
4	...	none	0

$h_{R/I}(t) = (1, 3, 3, 2)$

theorem (Macaulay)

Let $\underline{a} = (1, a_1, a_2, \ldots)$ be a sequence of non-negative integers. TFAE

(1) $\underline{a}$ is the Hilbert function of some standard graded k-algebra

(2) " " " " " " " monomial k-algebra

(3) $\underline{a}$ is an O-sequence, meaning each a_{i+1} is less than or equal to a very specific function of a_i (theorem 4.10 in notes)

Remarks

1. $\underline{a}$ is finite iff R/I is artinian

2. the notion of Hilbert functions extend to ~~grade~~ finitely generated graded modules

3. Let R/I be a standard graded k-algebra. Let L be a linear form

$\times L : [R/I](-1) \longrightarrow R/I$ is an R-module homomorphism.

in particular $\times L : [R/I]_{t-1} \longrightarrow [R/I]_t$ is a k-vector space homomorphism.

4. Given a homogeneous ideal $I \subset R$, the saturation of I is

$$I^{sat} = \{ f \in R \mid \text{for each } 1 \le i \le n \text{ there exists } a_i \text{ s.t. } x_i^{a_i} \cdot f \in I \}$$

I is saturated $\iff I = I^{sat}$

$\iff I : m = I$ for $m = (x_1, \dots, x_n)$.

5. Fact: if $X \subseteq \mathbb{P}^{n-1}$ is a subset then

$\mathbb{I}(X)$ = ideal generated by $\{ f \in R \mid f \text{ is homogeneous and } f(P) = 0 \ \forall P \in X \}$

is a saturated homog. ideal.

Ex. $X = \{ [0,0,1] \} \subseteq \mathbb{P}^2$, $I = \mathbb{I}(X) = (x,y) \subset k[x,y,z]$

$[I]_2 = \langle x^2, xy, xz, y^2, yz \rangle \subset [R]_2$

Let $J = (x^2, xy, xz, y^2, yz)$. J is not saturated

$J^{sat} = (x,y)$

6. depth $(R/I) \ge 1$ iff $\times L$ is injective in each degree

(Def 3.7)

7. If depth $(R/I) = 0$ there's no hope $\times L$ is always injective.

So the best we can hope for is $\times L$ is either injective or surjective

in each degree.

Def The algebra R/I has the Weak Lefschetz Property (WLP) if

$\times L : [R/I]_{t-1} \to [R/I]_t$ has maximal rank for all t, for a general

$L \in [R]_1$.

R/I has the <u>Strong Lefschetz Property</u> if

$$\times L^d : [R/I]_{t-d} \to [R/I]_t \text{ has max rank for all } t, \text{ all } d$$

<u>Remark</u> if depth $R/I \geq 1$ then R/I has WLP and SLP.

So we want to focus on depth $R/I = 0$.

<u>Remark</u> (Simplifications)

1. To prove WLP it's enough to find <u>one</u> $L \in [R]_1$ such that $\times L$ has max rank for all t.
2. if R/I is monomial, R/I has WLP iff $L = x_1 + \dots + x_n$ gives max rank
3. if R/I is Gorenstein (e.g. complete intersections) then R/I has WLP iff $\times L$ has max rank "in the middle"

<u>Example</u> $R = k[x,y,z]$ $\quad I = (x^3, y^3, z^3)$

Hilbert function $(1,3,6,7,6,3,1)$

R/I has WLP $\overset{\text{def}}{\iff}$ $\times L\, [R/I]_{t-1} \to [R/I]_t$ has max rank for general L and all t

$\iff$ $\times L : [R/I]_{t-1} \to [R/I]_t$ has max rank for one $L \in [R]_1$ and all t

$\iff$ - - - - - - - - - - - - - for $L = x+y+z$

$(*)$ $\iff$ $\times L : [R/I]_2 \to [R/I]_3$ is injective

$\iff$ $\times L : [R/I]_3 \to [R/I]_4$ is surjective

let $f = a_1 x^2 + a_2 xy + a_3 xz + a_4 y^2 + a_5 yz + a_6 z^2 \in [R/I]_2$

$Lf = 0 \iff (a_1, \dots, a_6) = (a_1, -a_1, -a_1, a_1, -a_1, a_1)$

and $3a_1 = 0$

injectivity iff char $k \neq 3$.

Remark one cheap way that $\times L : [R/I]_j \to [R/I]_{j+1}$ can fail to be injective is if R/I has a socle element in degree j.

$f \in [R/I]_j$ is a socle element if $f \cdot x_i = 0 \quad \forall i$ in R/I

i.e. $f \in \dfrac{I : m}{I} \qquad m = (x_1, \dots, x_n)$

Given f, $Lf = 0 \quad \forall L$ (socle)

given $L \quad \exists f$ s.t. $Lf = 0$ (failure of injectivity)

Lecture 1b

For WLP to be interesting we need depth $R/I = 0$

2 natural situations where this happens.

(Remark: both arise in the context of Jacobian ideals)

1. Let J be a saturated ideal and $I \subseteq J$ be an ideal for which $I^{sat} = J$.

Ex! J = ideal of 4 points in $\mathbb{P}^3$ in linear general position
$\leadsto \dim [J]_2 = 10 - 4 = 6$

I = ideal generated by a general 5-dimensional subspace of $[J]_2$.

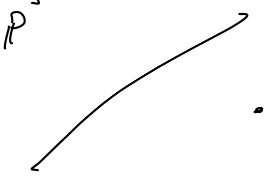
$h_{R/J} = (1,4,4,4,\ldots)$

$h_{R/I} = (1,4,5,4,4,\ldots)$

Let $f \in [J]_2 \setminus [I]_2$. f is a socle element

R/I has WLP

<u>Ex2</u> J = ideal of a line and a point in $\mathbb{P}^3$ $\qquad k=[x,y,z,t]$



$= (x,y) \cap (y,z,t)$

$h_{R/J} = (1,3,4,5,6,\ldots)$ $\qquad I = (xy, xz, xt, y^2, yz, yt)$

$h_{R/I} = (1,4,4,5,6,\ldots)$ $\qquad R/I$ fails WLP (from degree 1 to degree 2)

(2) R/I is <u>artinian</u>

<u>Def</u> A ring R/I is <u>Cohen-Macaulay</u> if depth R/I = Krull dimension of R/I.
A subvariety of $\mathbb{P}^{n-1}$ is arithmetically Cohen-Macaulay (ACM) if $R/I(X)$ is Cohen-Macaulay

R/I is artinian if depth R/I = krull-dim $R/I = 0$

$\Rightarrow R/I$ is CM.

<u>Minimal free resolutions</u> (MFR)

proj. dim (R/I) = length of MFR starting with 0.

<u>Ex</u> (a) $I = (x,y) \subset k[x,y,z,t]$ $\qquad$ (line in $\mathbb{P}^3$)

$0 \to R(-2) \to R(-1)^2 \to R \to R/I \to 0$ $\qquad$ proj dim = 2

(b) $I = (x^2, xy, y^2) \subset k[x, y, z, t]$ (fat line)

$$0 \to R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \to R/I \longrightarrow 0$$ proj dim 2

(c) $I = (xz, xt, yz, yt)$ (2 skew lines)

$$0 \to R(-4) \to R(-3)^4 \to R(-2)^4 \to R \to R/I \to 0$$ proj dim 3

Betti diagrams

(a)

	0	1	2
0	1	2	1
1	–	–	–

(b)

	0	1	2
0	1	–	–
1	–	3	2

(c)

	0	1	2	
0	1	–	–	–
1	–	4	4	1

<u>theorem</u> (Auslander - Buchsbaum) (special case)

let $R = k[x_1, \dots, x_n]$, $I \subset R$ homog ideal.

$$\text{proj dim}(R/I) + \text{depth}(R/I) = \text{depth}(R) = n$$

<u>Ex</u>

all three have krull dim 2, and $n = 4$

$$\left.\begin{array}{ll} \text{(a)} & \text{depth} = 4 - 2 = 2 \\ \text{(b)} & \text{depth} = 4 - 2 = 2 \end{array}\right\} \quad \text{CM}$$

(c) depth $R/I = 4-3 = 1$ not CM

<u>Ex</u> R/I artinian depth $=$ k-dim $= 0$

$\Rightarrow$ proj dim $= n - 0 = n$

Converse false

From now on assume artinian.

<u>Def</u> let R/I be artinian. An element $f \in R/I$ is a <u>socle element</u> if f is annihilated by $m = (x_1, -, x_n)$

$\Longleftrightarrow \quad f \in \dfrac{I:m}{I}$

The socle is the set of all socle elements

the socle can be read from the last column of the Betti table.

<u>Ex</u> $I = (x^3, y^4, z^5, x^2y^2, xyz^2)$

$h_{R/I} = (1, 3, 6, 9, 9, 6, 3, 1)$

	0	1	2	3
0	1	–	–	–
1	–	–	–	–
2	–	1	–	–
3	–	3	1	–
4	–	1	3	1
5	–	–	2	1
6	–	–	1	1
7	–	–	1	1

Gorenstein algebras

<u>Def</u> R/I is Gorenstein if it's CM and the rank of the last free module in the MFR is 1

in particular the socle is 1-dimensional.

<u>Def</u> if $X \subset \mathbb{P}^{n-1}$ is a subvariety and $R/I(X)$ is Gorenstein, we say X is <u>arithmetically Gorenstein</u>

<u>Ex</u>

1. $R = k[x,y,z]$ $\quad I = (x^2, y^3, z^3)$ $\quad$ HF $(1, 3, 5, 5, 3, 1)$

Betti diagram

	0	1	2	3
0	1	-	-	-
1	-	1	-	-
2	-	2	-	-
3	-	-	2	-
4	-	-	1	-
5	-	-	-	1

2. In general, complete intersections are Gorenstein

3. $X \subset \mathbb{P}^3$, $X = 5$ points in linear general position

$\Rightarrow X$ is arith Gor

the Hilbert function of an artinian Gorenstein algebra is symmetric

Special case – complete intersections

CI $\Rightarrow$ Gorenstein

For $n=2$, CI $\Leftrightarrow$ Gorenstein

Most important theorem for Lefschetz

<u>theorem</u> (Stanley, J. Watabe, Reid - Roberts - Roitman)

over a field of characteristic 0, any monomial complete intersection

$$R/(x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n}) \quad \text{has SLP}$$

Artinian reductions

If R/I is CM, but not artinian.

Say depth R/I = krull dim $R/I = d$

let L be a NZD (linear)

$\Rightarrow$ $R/(I, L)$ has depth = krull-dim $= d-1$

let $L_1, \dots, L_d$ general linear forms

$R/(I, L_1, \dots, L_d)$ is artinian

This is an artinian reduction of R/I.

The Hilbert function of the artinian reduction of R/I is the h-vector of R/I

Let R/I be CM, of depth = k-dim ≥ 1

Q1 what do you need from R/I for the general artinian reduction to have ~~SLP~~ WLP?

Q2 ~~what~~ when does every artinian reduction have WLP?