Lecture 15 (See notes for anything that's unclear)

From Lecture 14 we saw that in order for the Lefschetz properties to be interesting, we need $\operatorname{depth}(R/I) = 0$.

There are (at least) two natural ways to get this. BTW both arise in the context of Jacobian ideals (for example)

$\boxed{1.}$ Let J be a saturated homogeneous ideal and let $I \subset J$ be an ideal for which $I^{sat} = J$. E.g. $J = \mathbb{I}(X)$ for some $X \subset \mathbb{P}^{n-1}$.

This isn't that interesting

Ex1 J = ideal of 4 points in $\mathbb{P}^3$ - note $\dim [J]_2 = 10 - 4 = 6$

I = ideal generated by a general 5-dim'l subspace of $[J]_2$.

$h_{R/J} = (1, 4, 4, 4, \ldots)$

$h_{R/I} = (1, 4, 5, 4, 4, \ldots)$ because I "grows" to all of J in degrees ≥ 3. (Computer verified)

If $f \in [J]_2 \setminus [I_2]$ then f is a socle element of R/I.

Nevertheless R/I has WLP.

Ex2 J = ideal of a line union a point in $\mathbb{P}^3$

$= (x, y) \cap (y, z, t)$

$= (y, xz, xt) \subset k[x, y, z, t]$

$h_{R/J} = (1, 3, 4, 5, 6, \ldots)$

$I = (xy, xz, xt, y^2, yz, yt) \subset k[x, y, z, t]$

= ideal generated by $[J]_2$.

$h_{R/I} = (1, 4, 4, 5, 6, \ldots)$

R/I does not have WLP b/c if $\ell \in [R]_1$, $\ell y = 0$ so $\times \ell : [R/I]_1 \to [R/I]_2$ is not an isomorphism.

y is a socle element

Ex1 and Ex2 both have depth 0 but positive Krull dimension. (see Def 3.13 in notes)

Recall if $X \subset \mathbb{P}^{n-1}$ has dim d then the krull dimension is $d+1$.

$\boxed{2.}$ R/I is artinian. (Definition coming in a sec.)

Recall:

Def A ring R is Cohen-Macaulay if depth(R) = krull dimension of R.

A subvariety $X \subset \mathbb{P}^{n-1}$ is arithmetically Cohen-Macaulay (ACM) if $R/\mathbb{I}(X)$ is a Cohen-Macaulay ring.

Now: R/I is artinian if depth(R/I) = krull dim $(R/I) = 0$

(in particular artinian $\Rightarrow$ CM)

Minimal Free Resolutions

Recall proj. dim (R/I) = length of MFR (starting at 0)

Recurring

Ex(a) $I = (x,y) \subset k[x,y,z,t]$ (ideal of a line in $\mathbb{P}^3$). MFR:

$$0 \to R(-2) \to R(-1)^2 \to R \to R/I \to 0 \qquad \text{proj dim} = 2$$

(b) $I = (x^2, xy, y^2) \subset k[x,y,z,t]$ ("fat" line)

$$0 \to R(-3)^2 \to R(-2)^3 \to R \to R/I \to 0 \qquad \text{proj dim} = 2$$

(c) $I = (xz, xw, yz, yw)$ (two skew lines)

$$0 \to R(-4) \to R(-3)^4 \to R(-2)^4 \to R \to R/I \to 0 \qquad \text{proj dim} = 3$$

we'll see that being longer precludes this R/I from being CM.

Betti Diagrams

Ex (a)

	0	1	2
0	1	2	1
1	-	-	-

(b)

	0	1	2
0	1	-	-
1	-	3	2

(c)

	0	1	2	3
0	1	-	-	-
1	-	4	4	1

Theorem (Auslander - Buchsbaum) (Special case)

Let $R = k[x_1, \dots, x_n]$ and let $I \subset R$ be an ideal. Then

$$\operatorname{projdim}(R/I) + \operatorname{depth}(R/I) = \operatorname{depth} R = n$$

Ex In (a), (b), (c), all have Krull dimension 2, and $n = 4$

(a) has proj dim 2 $\Rightarrow$ depth $= 4 - 2 = 2$

$\Rightarrow R/I$ is CM

(b) has proj dim 2 $\Rightarrow R/I$ is CM

(c) has proj dim 3 $\Rightarrow \operatorname{depth}(R/I) = 4 - 3 = 1$

$\Rightarrow R/I$ is not CM.

Ex R/I artinian $\Rightarrow \operatorname{depth}(R/I) = \operatorname{Kdim}(R/I) = 0$

$\Rightarrow \operatorname{projdim}(R/I) = n - 0 = n$

Note converse is false. In fact (Auslander - Buchsbaum)

$\operatorname{projdim}(R/I) = n$ iff $\operatorname{depth}(R/I) = 0$ (but this doesn't force artinian).

From now on, unless specified otherwise we'll assume R/I is artinian (hence also CM) (so none of our examples applies)

Def Let R/I be artinian. An element $f \in R/I$ is a socle element

$\Leftrightarrow$ f is annihilated by $m = (x_1, -, x_n)$

$\Leftrightarrow$ $f \in \dfrac{I : m}{I}$

The socle is the set of all socle elements.

Very useful fact when R/I is artinian:

The socle of R/I can be read from the last column of the Betti table (or diagram) !

Ex $I = (x^3, y^4, z^5, x^2y^2, xyz^2)$

$h_{R/I}$: $(1,3,6,9,9,6,3,1)$

Betti diagram

	0	1	2	3
0	1	–	–	–
1	–	–	–	–
2	–	1	–	–
3	–	3	1	–
4	–	1	3	1
5	–	–	2	1
6	–	–	1	1
7	–	–	1	1

$\Rightarrow$ one-dim'l socle in degrees 4, 5, 6, 7

You'd expect that in degree 7 (why?) but not degrees 4, 5, 6.

Gorenstein Algebras

Def R/I is Gorenstein if it is CM and the rank of the last free module in the MFR is 1 (ie the last column adds up to 1)

In particular, if R/I is artinian Gorenstein then the socle of R/I is one-dimensional and occurs only at the end

e.g. $(1, 4, 8, 10, 11, 10, 8, 4, 1)$ Gor $\Rightarrow$ $\dim \operatorname{Soc}(R/I) = 1$ and it occurs in degree 8.

Def If $X \subset \mathbb{P}^n$ is a subvariety and $R/\mathbb{I}(X)$ is Gorenstein, we say X is arithmetically Gorenstein (AG) (Note Alexandra uses AG for "artinian Gorenstein")

Ex

1. $R = k[x,y,z]$, $I = (x^2, y^3, z^3)$ HF $(1, 3, 5, 5, 3, 1)$

Betti diagram

	0	1	2	3
0	1	–	–	–
1	–	1	–	–
2	–	2	–	–
3	–	–	2	–
4	–	–	1	–
5	–	–	–	1

note symmetry.

(note Gorenstein algebras are usually much less simple than this.)

Ex $X \subset \mathbb{P}^3$, $|X| = 5$

$\Rightarrow X$ is AG. (not artinian)

Remark The symmetry of the Betti diagram is related to the fact that R/I is self-dual. (Details omitted.) In particular, when R/I is artinian and Gorenstein, the Hilbert function is symmetric. More generally, Gorenstein of any dimension $\Rightarrow$ the h-vector is symmetric.

Special case : Complete intersections

R/I is a CI if (recalling artinian) the number of minimal generators of I is n.

Ex previous example.

Remark for any n, CI $\Rightarrow$ Gorenstein.

For $n=2$, CI $\Leftrightarrow$ Gorenstein

Most important (motivating) Theorem for Lefschetz properties:

Thm (Stanley, Watanabe, RRR)

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Over a field of char 0, any monomial CI $R/(x_1^{a_1}, \ldots, x_n^{a_n})$ has SLP.

Extra stuff if time:

$\boxed{\text{II.}}$ Artinian reductions

If R/I is CM but not artinian, we can always pass to the artinian reduction. More precisely:

If R/I is CM of depth = krull dimension = d and L is any non-zerodivisor then $R/(I, L)$ has krull dimension = depth = $d-1$ so $R/(I, L_1, \dots, L_d)$ (where all linear forms are general) is artinian.

Definition The Hilbert function of any artinian reduction of R/I by linear forms is called the h-vector of R/I.

Question We saw that for R/I CM of depth = k-dim ≥ 1 then WLP always holds. But how do you tell if

(a) a general artinian reduction of R/I has WLP?

(b) every artinian reduction of R/I has WLP?

Several papers have been written on this question.