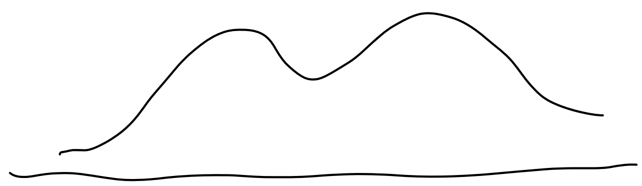
May 7, 2024

Lecture 2

Question If R/I has WLP, could its Hilbert function look like



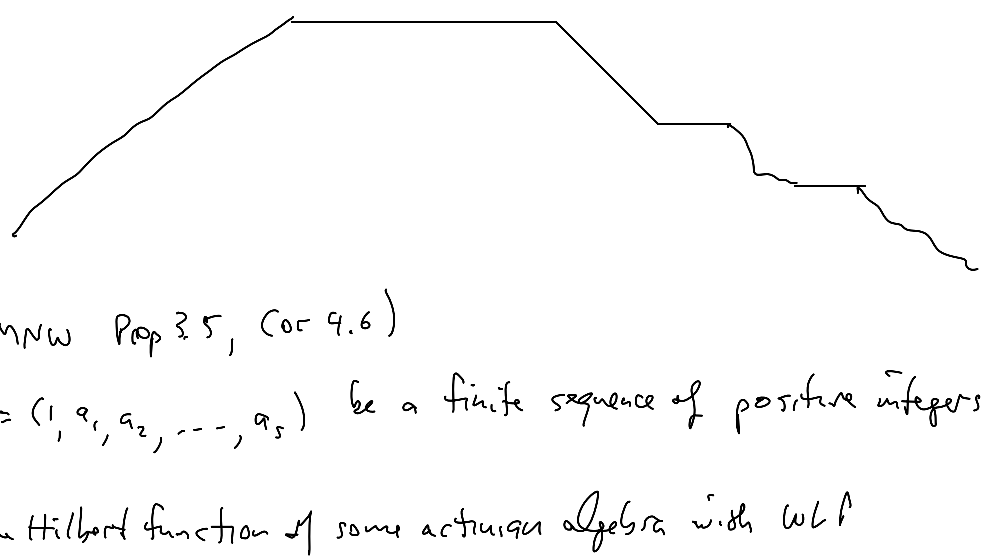
Lemma Let L be a linear form. We have the following exact sequence

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \frac{I:L}{I}(-1) \arrow[r] & R/I(-1) \arrow[rr, "\times L"] \arrow[dr] & & R/I \arrow[r] & R/(I,L) \arrow[r] & 0 \\
& & & R/I:L(-1) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Cor $0 \to R/I:L\,(-1) \xrightarrow{\times L} R/I \to R/I,L \to 0$

Cor once $\times L$ is surjective in some degree it's surjective thereafter.

In fact



Theorem (HMNW Prop 3.5, Cor 4.6)

Let $\underline{a} = (1, a_1, a_2, \dots, a_s)$ be a finite sequence of positive integers. TFAE

(1) $\underline{a}$ is the Hilbert function of some artinian algebra with WLP

(2) " " " " " " " " " " SLP

(3) $\underline{a}$ is unimodal and the positive part of the 1st difference

$$\Delta\underline{a} = (\dots, a_i - a_{i-1}, a_{i+1} - a_i, \dots)$$ is an $\mathcal{O}$-sequence

sketch the proof.

$$(2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (2)$$

$(2) \Rightarrow (1)$ obvious

$(1) \Rightarrow (3)$ Assume R/I has WLP, and Hilbert function $\underline{a} = (1, a_1, a_2, \dots, a_s)$.

Let L be a general linear form

$$0 \to \left[\frac{I:L}{I}(-1)\right]_{t-1} \to \left[R/I\right]_{t-1} \xrightarrow{\times L} \left[R/I\right]_t \to \left[R/(I,L)\right]_t \to 0$$

dimensions $\qquad a_{t-1} \qquad a_t$

By max rank

$$\dim \left[R/(I,L)\right]_t = \max\{0, a_t - a_{t-1}\}$$

$(3) \Rightarrow (2)$ See attached notes.

Ex Consider the HF

$$\underline{h} = (1, 4, 10, 15, 22, 22, 8, 8, 2)$$

this is an O-sequence.

the 1st difference:

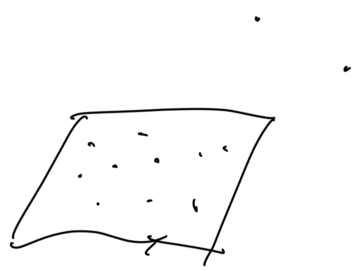
$$(1, 3, 6, 5, 7)$$

$$5 = \binom{4}{3} + \binom{2}{2} \longrightarrow \binom{5}{4} + \binom{3}{3} = 6$$

Ex (do artinian reductions have WLP always?)

Let $X \subset \mathbb{P}^4$ be a set of 12 consisting of 10 general points on a plane plus 2 general points in $\mathbb{P}^4$

$\mathbb{P}^4$



h-vectors

10 general points in a plane have h-vector $(1,2,3,4)$

HF of $R/I(X)$ is $(1,5,8,12,12 \dashrightarrow)$

$$0 \to [R/I(X)]_{t-1} \xrightarrow{L} [R/I(X)]_t \to [R/I(X),L]_t \to 0$$

HF $= (1,4,3,4)$

$\Rightarrow$ artinian reduction
can't have WLP

1 2 3 4

1 3 4 4

Questions

(a) do all artinian Gorenstein algebras have WLP?

(b) " " " CI's " "

(c) Monomial algebras?

(d) ideals generated by powers of linear forms

Talk about (a).

1. Stanley produced a Gorenstein example with HF $(1,13,12,13,1)$

Note $h_1 = 13$

2. For any $h_1 \geq 14$ there exist non-unimodal examples.

In the case of socle degree 4, what can we say for $h_1 \leq 12$?

<u>theorem</u> (M - Zanello) For socle degree 4, this is the smallest example

4. For $4 \leq h_1 \leq 12$ and socle degree 4, there exist non-WLP examples.

e.g. $(1,4,10,10,4,1)$ (Ikeda)

$(1,4,7,7,4,1)$ (Altafi - Dinu - Faridi - Masuti - Miró-Roig - Seceleanu - Villamizar)

5. (Boij) You can have lots of "valleys" in the HF

6. For arbitrary h_1 which Gorenstein Hilbert functions force WLP?

e.g. $(1,4,10,4,1)$, even socle degree

what about $(1,4,6,6,4,1)$?

7. Conjecture for $h_1 = 3$ All artinian Gorenstein algebras have WLP. (char 0)

8. for $h_1 = 2$ SLP always holds (char 0)

Lecture 26

(a) which artinian Gorenstein algebras have WLP?

(b) Do all CI's have WLP?

(c) Monomial algebras

(d) Ideals generated by powers of linear forms.

(c) Example (Exercises 51)

Brenner - Kaid 2007

$I = (x^3, y^3, z^3, xyz) \subset k[x,y,z] = R$

HF $(1,3,6,6,3)$ R/I is level but fails WLP.

(b) Complete intersections

For $n \le 3$, char 0, all CI's have WLP

For $n \ge 4$ much less is known but there are partial results.

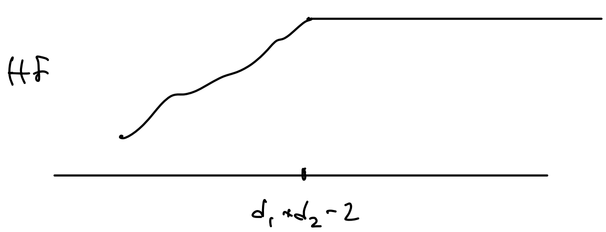
<u>Theorem</u> (HMNW)

Every CI in $k[x,y,z]$ over a field of char. 0 has WLP.

<u>sketch of proof</u>

Let (f_1, f_2, f_3) be a CI in $k[x,y,z]$. $\deg f_i = d_i$

- if $I = (f_1)$ or (f_1, f_2) — depth $R/I \ge 1$ so WLP holds.



- $I = (f_1, f_2, f_3)$ with $d_3 \ge d_1+d_2-1$ (e.g. $(d_1,d_2,d_3) = (2,3,7)$)

e.g. $(d_1,d_2,d_3) = (2,3,7)$ HF $= (1,3,5,6,6,6,6,5,3,1)$

injective in the middle

$\Rightarrow$ WLP

- $I = (f_1, f_2, f_3)$ with $d_3 \le d_1+d_2-2$

Let L be a general linear form.

WTS

$$[R/I]_{t-1} \xrightarrow{\times L} [R/I]_t \quad \text{has max rank } \forall t.$$

Let $\mathbb{F} = R(-d_1) \oplus R(-d_2) \oplus R(-d_3)$

$\mathbb{G} = R(-d_1-d_2) \oplus R(-d_1-d_3) \oplus R(-d_2-d_3)$

Koszul resolution

$$
\begin{tikzcd}
0 \arrow[r] & R(-d_1-d_2-d_3) \arrow[r] & \mathbb{G} \arrow[rr] \arrow[dr] & & \mathbb{F} \arrow[r] & R \arrow[r] & R/I \arrow[r] & 0 \\
& & & E \arrow[ur] & & & & \\
& & 0 \arrow[ur] & & & & &
\end{tikzcd}
$$

E is the syzygy module

$\Rightarrow$ $0 \to R(-d_1-d_2-d_3) \to \mathbb{G} \to E \to 0$

$0 \to E \to \mathbb{F} \to R \to R/I \to 0$

Commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & E(-1) \arrow[r] & \mathbb{F}(-1) \arrow[r] \arrow[d, "{\begin{bmatrix} h & & 0 \\ & h & \\ 0 & & h \end{bmatrix}}"] & R(-1) \arrow[r] \arrow[d, "\times h"] & R/I(-1) \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathbb{F} \arrow[r] & R \arrow[r] & R/I \arrow[r] & 0
\end{tikzcd}
$$

Let $\overline{R} = R/(h) \cong k[x,y]$

$$
\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E(-1) \arrow[r] & \mathbb{F}(-1) \arrow[r] \arrow[d, "{\begin{bmatrix} h & & 0 \\ & h & \\ 0 & & h \end{bmatrix}}"] & R(-1) \arrow[r] \arrow[d, "\times h"] & R/I(-1) \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/I \arrow[r] & 0 \\
& & \overline{\mathbb{F}} \arrow[r] \arrow[d] & \overline{R} \arrow[d] & & \\
& & 0 & 0 & &
\end{tikzcd}
$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E(-1) \arrow[r] & \mathbb{F}(-1) \arrow[r] \arrow[d, "{\begin{bmatrix} L & & 0 \\ & L & \\ 0 & & L \end{bmatrix}}"] & R(-1) \arrow[r] \arrow[d, "\times L"] & R/I(-1) \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/I \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] & \overline{\mathbb{F}} \arrow[r] \arrow[d] & \overline{R} \arrow[r] \arrow[d] & B \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

Snake Lemma

$$0 \to E(-1) \to E \to A \to R/I(-1) \to R/I \to B \to 0$$

Look in degree t.

$$0 \to [E(-1)]_t \to [E]_t \to [A]_t \to [R/I(-1)]_t \xrightarrow{\times L} [R/I]_t \to [B]_t \to 0$$

$(*)$ it's enough to show that for all t, either $[A]_t = 0$ or $[B]_t = 0$

Now sheafify. Let

$$\mathcal{E} = \widetilde{E}$$

$$\mathcal{F} = \widetilde{\mathbb{F}}$$

$$\overline{\mathcal{F}} = \widetilde{(\overline{\mathbb{F}})}$$

$$\mathcal{O}_{\mathbb{P}^2} = \widetilde{R}$$

$$\mathcal{O}_{\mathbb{P}^1} = \widetilde{(\overline{R})}$$

$$\mathcal{A} = \widetilde{A}$$

$$\widetilde{R/I} = 0, \quad \widetilde{B} = 0$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}(-1) \arrow[r] & \mathcal{F}(-1) \arrow[r] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r] & \tilde{\mathcal{F}} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Snake lemma gives maps

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}(-1) \arrow[r] \arrow[d] & \mathcal{F}(-1) \arrow[r] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \tilde{\mathcal{F}} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\Rightarrow \quad \mathcal{A} = \mathcal{E}|_{\mathbb{P}^1}$

For $d_3 \leq d_1 + d_2 - 2$, $\mathcal{E}$ is semistable

$\Rightarrow$ (Grauert-Mülich) $\quad \underset{\mathcal{A}}{\underset{\|}{\mathcal{E}|_{\mathbb{P}^1}}} = \mathcal{O}_{\mathbb{P}}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$ with $|a-b| \leq 1$

$\Rightarrow$ $(*)$ holds

$$h^0(\mathcal{O}_{\mathbb{P}}(t)) = \begin{cases} 0 & t \leq -1 \\ t+1 & t \geq 0 \end{cases}$$

$$h^1(\mathcal{O}_{\mathbb{P}^1}(t)) = \begin{cases} 0 & t \geq -1 \\ -t+3 & t \leq -2 \end{cases}$$

Jacobian ideals

e.g. $F \in k[x_1, x_2, x_3, x_4]$

$$J = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_4}\right)$$

when F is smooth, R/J is an artinian CI

when F is not smooth, J defines some subscheme of $\mathbb{P}^3$

but J may or may not be saturated

If J is saturated, R/J may or may not be CM.

Special case: F is a product of linear forms.