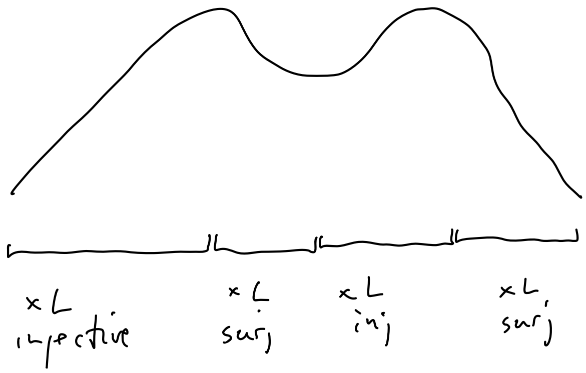
Lecture 29

Now let's start focusing on the Lefschetz properties. (Add to what Alexandra told us.)

Start with the HF. What can be its shape?

Question: if R/I has the WLP, could its HF have the following shape?



No! In fact we have a complete description of the possible HFs of algebras with the WLP or SLP.

First a useful exact sequence:

Lemma Let L be a linear form. The following sequence of graded R-modules is exact and degree zero

$$0 \to \frac{I:L}{I}(-1) \to R/I\,(-1) \xrightarrow{\times L} R/I \to R/(I,L) \to 0$$

Cor

$$0 \to R/(I:L)\,(-1) \xrightarrow{\times L} R/I \to R/(I,L) \to 0$$

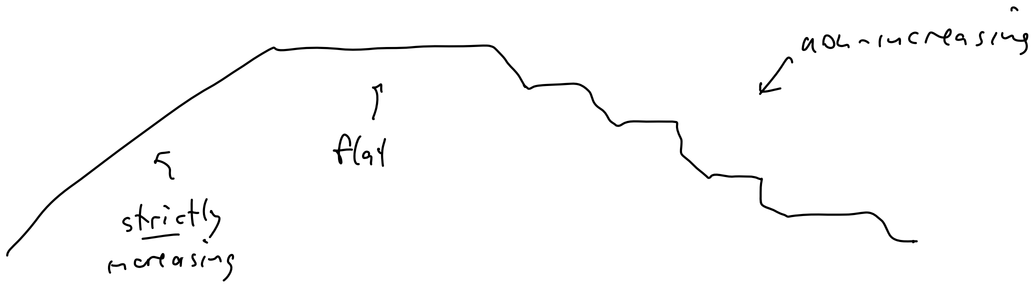
Cor Once $\times L : [R/I]_{t-1} \to [R/I]_t$ is surjective, it's surjective for all $s \geq t$ since once $[R/(I,L)]_t = 0$, it's 0 $\forall s \geq t$. (Standard graded algebra) So the above picture is impossible for WLP.

Here's a more accurate picture:

Remark (Imprecise) If R/I is artinian and has WLP then its Hilbert function is unimodal. In fact:

say, don't write

- the 1st part is strictly increasing, and the 1st difference of the HF for this part is an $\mathcal{O}$-sequence.
- Then the HF possibly is flat for a while at the highest value (called the Sperner number)
- Then the HF is non-increasing but could be flat in places.



More precisely:

thm (HMNW Prop 3.5, Cor 4.6)

Let $\underline{a} = (1, a_1, \dots, a_s)$ be a finite sequence of positive integers. TFAE

(1) $\underline{a}$ is the HF of some artinian algebra with the WLP.

(2) $\underline{a}$ " " " " " " " " SLP.

(3) $\underline{a}$ is unimodal and the positive part of the first difference

$$\Delta \underline{a} = (\dots a_i - a_{i-1}, a_{i+1} - a_i, \dots)$$

is an $\mathcal{O}$-sequence.

sketch of proof (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2)

(2) $\Rightarrow$ (1) obvious

(1) $\Rightarrow$ (3)

Assume R/I has WLP and HF $(1, a_1, \dots, a_s)$. Let L be a general linear form.

Consider for $1 \le t \le s+1$

$$0 \to \left[\frac{I:L}{I}\right]_{t-1} \to \underset{a_{t-1}}{\left[R/I\right]_{t-1}} \xrightarrow{\times L} \underset{a_t}{\left[R/I\right]_t} \to \left[R/(I,L)\right]_t \to 0$$

dim:

By max rank (WLP)

$$\dim\left[R/(I,L)\right]_t = \max\{0, a_t - a_{t-1}\}$$

So the statement of (3) holds.

(3) $\Rightarrow$ (2) See HMNW Prop 3.5 and Cor 4.6 (attached to notes). It uses a specific construction that we'll omit here.

In particular it explains why the mysterious "flats" are possible at the end of the HF.

<u>Example</u> Consider the HF

$\underline{h} = (1, 4, 10, 15, 22, 22, 8, 8, 2)$

You can check it is an O-sequence. In particular,

$$15 = \binom{5}{3} + \binom{3}{2} + \binom{2}{1} \to \binom{6}{4} + \binom{4}{3} + \binom{3}{2} = 15 + 4 + 3 = 22$$

But the 1st difference (positive part) is

1 3 6 5 7

and $5 = \binom{4}{3} + \binom{2}{2} \to \binom{5}{4} + \binom{3}{3} = 6$

So the 1st difference is not an O-sequence.

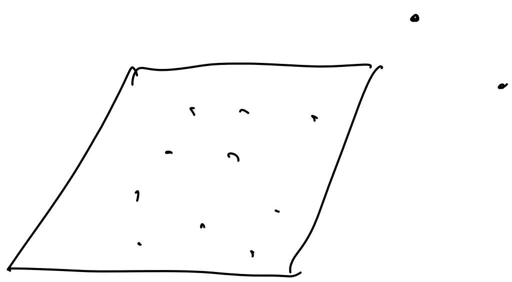
Conclusion: even though $\underline{h}$ is the Hilbert function of some artinian algebra, and $\underline{h}$ is unimodal, the algebra can't have WLP.

Remark HMNW also showed that the constructed algebra has maximal Betti numbers among algebras with the WLP.

Last time:

How do we know if an artinian reduction of R/I has WLP?

Example Let $X \subset \mathbb{P}^4$ be a set of 12 pts consisting of 10 general pts on a plane, plus 2 general pts. in $\mathbb{P}^4$



1 4 3 4

(a) Quick argument for Hilbert function:

$$\boxed{\begin{array}{l} h_{R/I(X)}(0) = 1 \\ h_{R/I(X)}(1) = 5 \end{array}}$$

the plane is a complete intersection of linear forms, so the plane lies on 7 quadrics $\therefore$ $\boxed{h_{R/I(X)}(2) = 15 - 7 = 8}$

Then check X imposes indep cond on forms of degree ≥ 3

So $\dim [I(X)]_t = \binom{t+4}{4} - 12$

in partic $\boxed{h_{R/I(X)}(3) = 12 \ , \ h_{R/I(X)}(4) = 12 \ ----}$

X is ACM of k-dim 1 so depth $= 1$.

Let L be a gen'l linear form. Then $\times L : [R/I(X)]_{t-1} \to [R/I(X)]_t$ is injective $\forall t$.

But what about the artinian reduction? $A = R/(I, L)$.

Consider the sequence

$$0 \to [R/I(X)]_{t-1} \xrightarrow{\times L} [R/I(X)]_t \longrightarrow [R/(I(X), L)]_t \longrightarrow 0$$

$\Rightarrow$ HF of A is 1st difference of HF of X

$(1, 4, 3, 4)$

$\therefore$ A does not have WLP!

Some important areas of research on WLP:

(a) do all artinian Gorenstein algebras have WLP? (any # of variables)

(b) do all CI's have WLP? (any # of variables)

(c) Monomial algebras

(d) Ideals generated by powers of linear forms

Let's talk about (a) first.

1. Stanley showed that $\exists$ Gorenstein algebra with HF $(1,13,12,13,1)$.
This is not unimodal, hence does not have WLP. Note that since $h_1=13$, this algebra involves 13 variables.

2. For any $h_1 \geq 14$ there also exists a non-unimodal $(1,h_1,h_2,h_1,1)$.
So the question is what can we say for $h_1 \leq 12$?
Theorem (M-Z): For socle degree 4, $h_1=13$ is the smallest for which a nonunimodal example exists.

3. Unfortunately, for $4 \leq h_1 \leq 12$ there exist non-WLP examples (with unimodal HF). (Any char.)
Some examples: $(1,4,10,10,4,1)$ (Ikeda) (even compressed)
$(1,4,7,7,4,1)$ (Altafi - Dinu - Faridi - Masuti - MR - Seceleanu - Villamizar)

For arbitrary h_1 which HFs force WLP (for Gorenstein)?

e.g. compressed, even socle degree like $(1,4,10,4,1)$

Char 0:
For $h_1=4$, $h_2 \leq 6$ WLP and even SLP are forced (BMMN in progress)
BMMN in progress also gives new situation (char 0) for arb. h_1.

4. Conjecture: for $h_1=3$, WLP always holds (char 0)
(Some work by BMMNZ among others)

5. For $h_1=2$, char 0, SLP always holds.