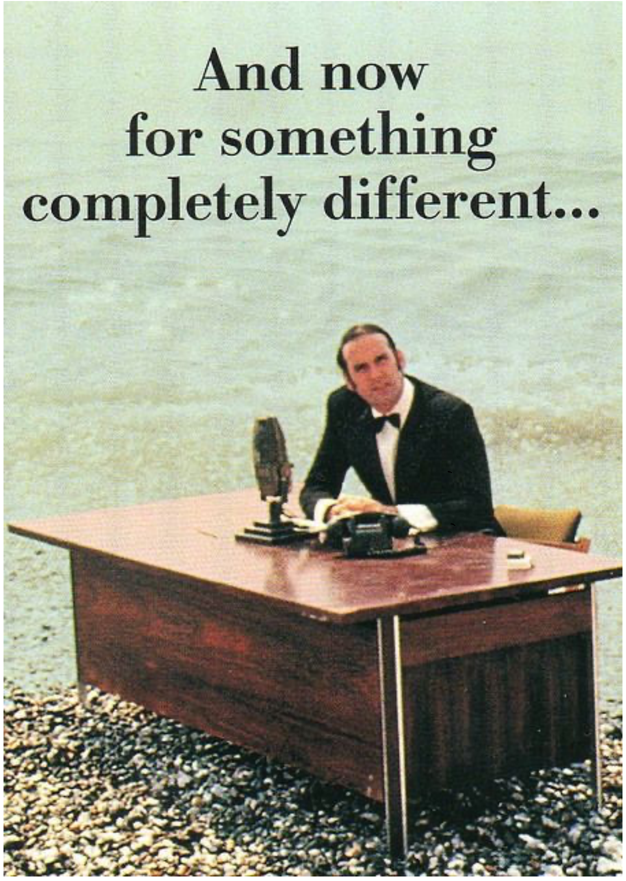
Lecture 3a



I. The non-Lefschetz locus

Def Let R/I be a standard graded artinian algebra. The non-Lefschetz locus $\mathcal{L}_{R/I}$ is the set of $L \in [R]_1$ that are not Lefschetz elements.

Cleanest situation: R/I is Gorenstein

Lemma Let R/I be graded artinian Gorenstein with Hilbert function either

(a) $$(1, r, h_2, \dots, h_t, \underset{h_t}{\underset{\shortparallel}{h_{t+1}}}, \dots, \underset{h_2}{\underset{\shortparallel}{h_{2t-1}}}, \underset{r}{\underset{\shortparallel}{h_{2t}}}, \underset{1}{\underset{\shortparallel}{h_{2t+1}}})$$ or

(b) $$(1, r, h_2, \dots, h_{t-1}, h_t, \underset{h_{t-1}}{\underset{\shortparallel}{h_{t+1}}}, \dots, \underset{h_2}{\underset{\shortparallel}{h_{2t-2}}}, \underset{r}{\underset{\shortparallel}{h_{2t-1}}}, \underset{1}{\underset{\shortparallel}{h_{2t}}})$$

Let L be a linear form. Let $i \geq t$

If $\times L : [R/I]_i \to [R/I]_{i+1}$ fails to be surjective then

$\times L : [R/I]_t \to [R/I]_{t+1}$ " " " "

So you can find $\mathcal{L}_{R/I}$ by looking at

Remarks

1. Injectivity comes for free since R/I is Gorenstein
2. There's a stronger ideal-theoretic statement
3. Leave proof to you.

Ex $\quad I = (x^2, y^3, z^4) \subset k[x,y,z]$ $\quad$ Let $L = ax+by+cz$

HF $(1,3,5,6,5,3,1)$

Choose bases for $[R/I]_3$ and $[R/I]_4$. $\quad$ ~~Let~~ $\quad$ $(ax+by+cz)(xy^2)$

$$\begin{array}{c|cccccc} & xy^2 & xyz & xz^2 & y^2z & yz^2 & z^3 \\ \hline xy^2z & c & b & 0 & a & 0 & 0 \\ xyz^2 & 0 & c & b & 0 & a & 0 \\ xz^3 & 0 & 0 & c & 0 & 0 & a \\ y^2z^2 & 0 & 0 & 0 & c & b & 0 \\ yz^3 & 0 & 0 & 0 & 0 & c & b \end{array}$$

As a subscheme, $\mathcal{L}_{R/I} \subseteq \mathbb{P}^2$ is defined by the ideal of maximal minors of this matrix. It defines a nonreduced curve of degree 2 in $\mathbb{P}^2$ with Hilbert polynomial $2t+5$.

II Linear Systems and independent conditions

Let $R = k[x,y,z]$ and consider $[R]_2$.

basis: $x^2, xy, xz, y^2, yz, z^2$

Typical element $a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2$

- two elements of $[R]_2$ define the same curve in $\mathbb{P}^2$ (as schemes) iff they differ by a scalar multiple.

  $\Rightarrow$ think of $\mathbb{P}[R]_2 = \mathbb{P}^5$ as parametrizing curves in $\mathbb{P}^2$ of degree 2.

- Let $P = [1,2,3] \in \mathbb{P}^2$

  Which curves vanish at P?

  $$a_1(1)^2 + a_2(1)(2) + a_3(1)(3) + a_4(2)^2 + a_5(2)(3) + a_6(3)^2 = 0$$

  $$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_5 + 9a_6 = 0$$

  this defines a 5-dim'l vector space in $[R]_2$

  $\Rightarrow$ hyperplane in $\mathbb{P}^5$

- P imposes one condition on conics in $\mathbb{P}^2$

- If there are k points we get k homogeneous linear equation in $a_1, \dots, a_6$

  We say the k points impose independent condition if the corresp. linear equations are linearly independent.

$\rightsquigarrow \mathbb{P}^n$, $R = k[x_0, \dots, x_n]$

- A linear system of hypersurfaces of degree d in $\mathbb{P}^n$ is the projectivization of a vector subspace of $[R]_d$.

- $\dim [R] = \binom{n+d}{n}$ so $\{\text{hypersurfaces in } \mathbb{P}^n \text{ of degree } d\} = \mathbb{P}^{\binom{n+d}{n}-1}$

- A general set of k points imposes

  $$\min\left\{ k, \binom{n+d}{n} \right\} \text{ conditions}$$

- Question: How many conditions on $\mathbb{P}^{\binom{n+d}{n}-1}$ is it for a hypersurface of degree d to be singular at one point P? (ie vanish to multiplicity 2)

Answer: we need the $n+1$ first partial derivatives to vanish at P.

$n+1$ conditions

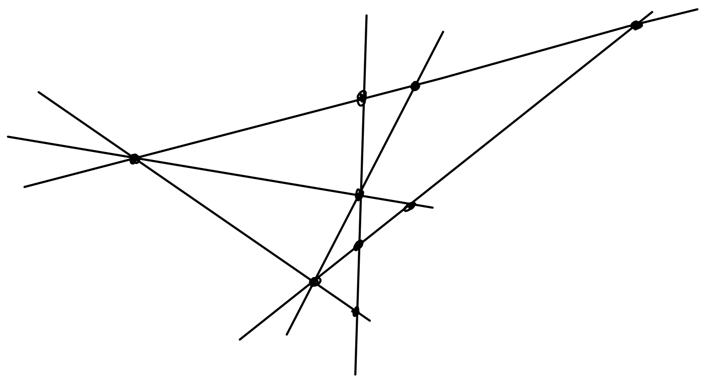
- to vanish at P to multiplicity r we want the $(r-1)^{st}$ partials to vanish

$\binom{r-1+n}{n}$ conditions.

<u>Fact</u> these are independent conditions (as long as there's room)

III. Unexpected hypersurfaces

<u>Example</u> (Di Gennaro – Ilardi – Valles)



Let Z be these 9 points.

HF $(1, 3, 6, 9, 9, \ldots)$

$\Rightarrow \dim [I_Z]_4 = 15 - 9 = 6 \leftarrow$ vector space

$\Rightarrow$ linear system $\mathbb{P}[I_Z]_4 = \mathbb{P}^5$

Let P be a general point. We expect a general point, vanishing to order 3, to impose $\binom{3-1+2}{2} = 6$ conditions

So we "expect" that no elements of $[I_Z]_4$ vanish to order 3 at a general point.

And yet there always is one!

This example launched the study of unexpected hypersurfaces.

More precisely

$F \in [I_Z]_d$ vanishes to order r at P

iff $F \in [I_Z \cap I_P^r]_d$

Def Given $Z \subset \mathbb{P}^n$, positive integers d, r, general point P

the <u>actual dimension</u> is $a\text{-dim} = \dim [I_Z \cap I_P^r]_d$

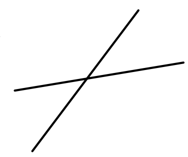
the virtual dimension is $v\text{-dim} = \dim [I_Z]_d - \binom{r-1+n}{n}$

the expected dimension is $e\text{-dim} = \max\{v\text{-dim}, 0\}$

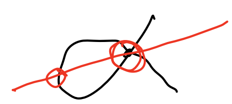
Z <u>admits an unexpected hypersurface of deg d and multiplicity r</u> if

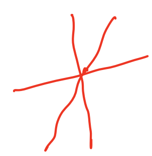
$$a\text{-dim} > e\text{-dim}$$

d=2



d=3



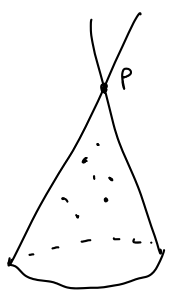


IV Unexpected Cones

Def in the previous definition, if $d = r$

then we say Z admits an unexpected cone of degree d.

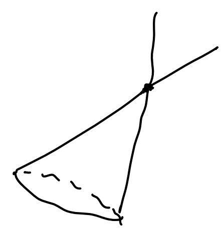
Important Observation



$Z \subset C$ = quadric cone

projection from P

$= \pi_P(Z)$ lies on a plane conic



Lecture 36

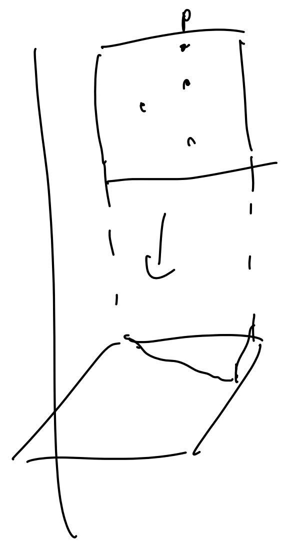
Weddle Surfaces

Let $Z \subset \mathbb{P}^3$ be a set of 6 points in LGP

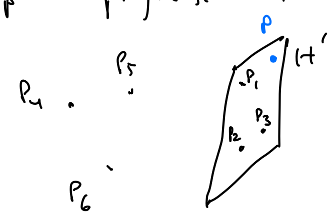
HF $(1, 4, 6, 6, \ldots)$

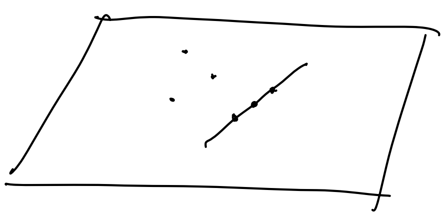
Let $Z = \{P_1, \ldots, P_6\}$

Claim A general projection of Z to $\mathbb{P}^2$ does not lie on a conic.



- it suffices to find one such projection
- Pick any 3 of the points. They span a plane H'.
- the other 3 points of Z are off H'. (by LGP)
- Let P be a general point of H'. Let π_p be projection from P.





$\Rightarrow$ $\pi_p(Z)$ does not lie on a conic

Done the claim

Question Which points $P \in \mathbb{P}^3$ do have the property that $\pi_p(Z)$ lies on a conic?

Def Let W be the set of all such points. W is the Weddle surface.

Algebraic observation

TFAE

(1) $P \in W$

(2) $\pi_P(Z)$ lie on a conic

(3) Z lies on a quadric cone with vertex P

(4) $[I_Z \cap I_P^2]_2 \neq 0$

What does W look like?

Some "obvious" subsets of W

1. Let $\overline{P_iP_j}$ be the line joining P_i, P_j. The projection of Z from a point $P \in \overline{P_iP_j}$ lies on a conic so $\overline{P_iP_j} \subseteq W$

   $\Rightarrow$ get 15 lines on W

2. Partition Z into 2 sets of 3 points, say $\{P_1, P_2, P_3\} \cup \{P_4, P_5, P_6\}$

   Get 2 planes H_1, H_2. Let $\lambda = H_1 \cap H_2$.
   Then projecting from any $P \in \lambda$ sends Z to 6 points on a reducible conic.

   Get 10 more lines.

3. the points of Z lie on W.

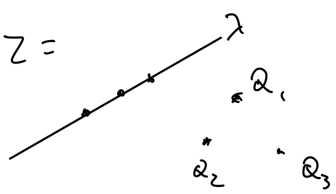
Facts (prove in a moment)

1. W is a irreducible surface of degree 4

2. W is the non-Lefschetz locus of a certain artinian graded algebra with WLP

3. if we tweak LGP a bit we get different behavior

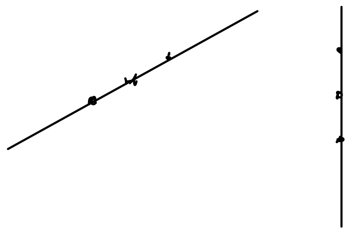
   e.g. $Z =$



$\Rightarrow$ $\mathcal{W}$ is a union of 4 planes

$\overline{Q_1 Q_2 Q_3}$, $\overline{\lambda Q_1}$, $\overline{\lambda Q_2}$, $\overline{\lambda Q_3}$

4. More extreme:



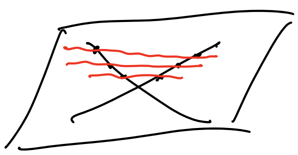
$\mathcal{W}$ is all of $\mathbb{P}^3$!

The corresponding algebra fails WLP!

5. Even: these questions go way beyond sets of 6 points

6. Look at the example from fact 4

$\downarrow$



the image points are a complete intersection of type (2,3)

Q: What can you say about sets in $\mathbb{P}^3$ whose general projection is a complete intersection?

geproci = GEneral PROjection is a Complete Intersection.

Connection to Lefschetz (6 points in LGP)

$Z = \{P_1, \dots, P_6\} \subset \mathbb{P}^3$ in LGP.

$\mathcal{W}$ = set of points P from which $\pi_P(Z)$ lies on a conic

$R = k[x,y,z,w]$

$\wp_1 = \mathbb{I}(P_1)$

$\vdots$

$\wp_6 = \mathbb{I}(P_6)$

$p = I(P)$ (P is the point from which we project)

Each point in $\mathbb{P}^3$ is dual to a linear form

$$P = [a,b,c,d] \longleftrightarrow ax+by+cz+dw \in [R]_1$$

$$P_1 \longleftrightarrow \ell_1$$
$$\vdots$$
$$P_6 \longleftrightarrow \ell_6$$
$$P \longleftrightarrow \ell$$

~~[illegible]~~

<u>Question</u> for which a,b,c,d is it true that for $P=[a,b,c,d]$,

$$\dim [p_1 \cap \cdots \cap p_6 \cap p^2]_2 \neq 0$$

$$\dim [p_1 \cap \cdots \cap p_6 \cap p^2]_2 = \dim [R/(\ell_1^2, \ldots, \ell_6^2, \ell)]_2$$

$$\underset{4}{[R/(\ell_1^2,\ldots,\ell_6^2)]_1} \xrightarrow{\times \ell} \underset{4}{[R/(\ell_1^2,\ldots,\ell_6^2)]_2} \longrightarrow [R/(\ell_1^2,\ldots,\ell_6^2,\ell)]_2 \to 0$$

Translation: which ℓ make $[R/(\ell_1^2,\ldots,\ell_6^2,\ell)]_2 \neq 0$

" " " $\times \ell$ not surjective ?

(in ~~general~~ principle this could be all ℓ

$\Rightarrow R/(\ell_1^2,\ldots,\ell_6^2)$ fails WLP)

Such ℓ give the non-Lefschetz locus

represent $\times \ell$ by a matrix.

1st Question: what's the size of the matrix ?

' 4 6 →

$$\dim [R/(\ell_1^2,\ldots,\ell_6^2)]_1 = 4$$

$$\dim [R/(\ell_1^2,\ldots,\ell_6^2)]_2 = \dim [p_1 \cap \cdots \cap p_6]_2 = \dim [I(Z)]_2$$

$$= \dim [R]_2 - |Z| = 10-6=4$$

So A is a 4×4 matrix.

2. Entries of A.

Choose bases for $[R/(\ell_1^2, \ldots, \ell_6^2)]_1$ and $[R/(\ell_1^2, \ldots, \ell_6^2)]_2$

$$\ell = ax + by + cz + dw$$

$\cdot x \longleftrightarrow A_1$

$\cdot y \longleftrightarrow A_2$

$\cdot z \longleftrightarrow A_3$

$\cdot w \longleftrightarrow A_4$

$\cdot \ell \longleftrightarrow aA_1 + bA_2 + cA_3 + dA_4$

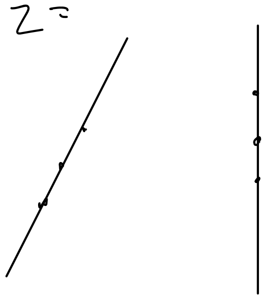
$=$ 4×4 matrix of linear forms in a, b, c, d.

$\det A$ gives $\{\ell$ for which $\cdot \ell$ fails max rank$\}$

"$=$" Weddle surface

$\Rightarrow$ $\mathcal{W}$ is both the non-Lefschetz locus and the Weddle surface for Z.

$Z =$



HF is the same

1 4 6 6 ...

$\Rightarrow$ $\mathcal{W} = \mathbb{P}^3$

and $R/(\ell_1^2, \ldots, \ell_6^2)$ fails WLP!