Lecture 34 **Offshoots** (i.e. related topics)

I. The non-Lefschetz locus (scheme)

Def Let R/I be a standard graded artinian algebra. The non-Lefschetz locus $\mathcal{L}_{R/I}$ is the set of $L \in [R]_1$ that are not Lefschetz elements.

The cleanest situation is when R/I is Gorenstein.

Lemma Let R/I be a graded artinian Gorenstein algebra with Hilbert fctn either

(a) $(1, r, h_2, \ldots, h_t, h_{t+1} \ldots h_{2t-1}\ h_{2t}\ h_{2t+1})$ (i.e. odd socle degree $2t+1$), or

$$h_{t+1} = h_t, \ldots, h_{2t-1} = h_2, h_{2t} = r, h_{2t+1} = 1$$

(b) $(1, r, h_2, \ldots, h_{t-1}, h_t, h_{t+1}, \ldots, h_{2t-2}\ h_{2t-1}\ h_{2t}$ (i.e. even socle degree $2t$).

$$h_{t+1} = h_{t-1}, \ldots, h_{2t-2} = h_2, h_{2t-1} = r, h_{2t} = 1$$

Let L be a linear form, let $i \geq t$.

If $\times L : [R/I]_i \longrightarrow [R/I]_{i+1}$ fails to be surjective then

$\times L : [R/I]_t \to [R/I]_{t+1}$ " " " " .

So to capture all the non-Lefschetz elements, it's enough to look at

$\times L : [R/I]_t \longrightarrow [R/I]_{t+1}$ (i.e "in the middle")

Remarks
1. Duality gives a statement about injectivity (important!)
2. $\exists$ a stronger ideal-theoretic statement that we'll omit.
3. I'll leave the proof to you

How do we get our hands on $\mathcal{L}_{R/I}$ when R/I is Gorenstein?

$\underline{\text{Ex}}$ $\quad I = (x^2, y^3, z^4) \subset k[x,y,z]$

HF $\quad (1, 3, 5, \underbrace{6,}_{\text{middle}} 5, 3, 1)$ $\qquad$ let $L = ax + by + cz$

Choose a basis for $[R/I]_3$, $[R/I]_4$; and form the matrix for $\times L$

$$\begin{array}{c|cccccc} & xy^2 & xyz & xz^2 & y^2z & yz^2 & z^3 \\ \hline xy^2z & c & b & 0 & a & 0 & 0 \\ xyz^2 & 0 & c & b & 0 & a & 0 \\ xz^3 & 0 & 0 & c & 0 & 0 & a \\ y^2z^2 & 0 & 0 & 0 & c & b & 0 \\ yz^3 & 0 & 0 & 0 & 0 & c & b \end{array}$$

$\mathcal{L}_{R/I}$ is given by the ideal of maximal minors of this matrix! (It defines a non-reduced curve of degree 2 in $\mathbb{P}^2$ w/ variables a, b, c) $HP = 2t + 5$.

II. Linear Systems + Independent Conditions

Let $R = k[x,y,z]$ and consider the v.s. $[R]_2$.

Basis: $\quad x^2, xy, xz, y^2, yz, z^2$

typical element $\quad a_1 x^2 + \cdots + a_6 z^2$

- Two elements of $[R]_2$ define the same curve in $\mathbb{P}^2$ (as schemes) iff they differ by a scalar multiple.

  $\Rightarrow$ think of $\mathbb{P}[R]_2 = \mathbb{P}^5$ as parametrizing the curves in $\mathbb{P}^2$ of degree 2.

- Let $P = [1,2,3] \in \mathbb{P}^2$.

  which curves in $\mathbb{P}^2$ contain P?

$$a_1(1)^2 + a_2(1)(2) + a_3(1)(3) + a_4(2)^2 + a_5(2)(3) + a_6(3)^2 = 0$$

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_5 + 9a_6 = 0$$

This defines a 5-dimensional vector subspace of $[R]_2$.

Thus it defines a <u>hyperplane</u> in $\mathbb{P}^5$, ie 4-dim'l linear space inside $\mathbb{P}^5$

- We say P <u>imposes</u> <u>one</u> <u>condition</u> on conics in $\mathbb{P}^2$ since the solution space has codim 1 : it's the solution space of one (independent) homog. linear equation.

- If there are k points we get k homog. linear equations, which may or may not be independent.
  We say the k points <u>impose independent conditions</u> if the corresp equations are independent.

Move to $\mathbb{P}^n$.

- In general, a <u>linear</u> <u>system</u> in $\mathbb{P}^n$ is the projectivization of a vector subspace of $[R]_d$ for some d, where here $R = k[x_0, -, x_n]$.

- Recall $[R]_d$ is a v.s. of dim $\binom{n+d}{n}$ and thus the projectivization is a projective space $\mathbb{P}^{\binom{n+d}{n}-1}$ of hypersurfaces of degree d.

- A <u>general</u> set of k points imposes $\min\{k, \binom{d+n}{n}\}$ conditions.

<u>Fact</u> For <u>any</u> linear system that's at least 1-dim'l, imposing the vanishing at a <u>general</u> point is one new condition.

Now go one step farther.

- Question: how many conditions on $\mathbb{P}^{\binom{d+n}{n}-1}$ is it for a hypersurface of degree d to be singular at a point P, i.e. vanish to multiplicity 2?

Answer: we want the $n+1$ first partial derivatives to vanish at P, so $n+1$ conditions.

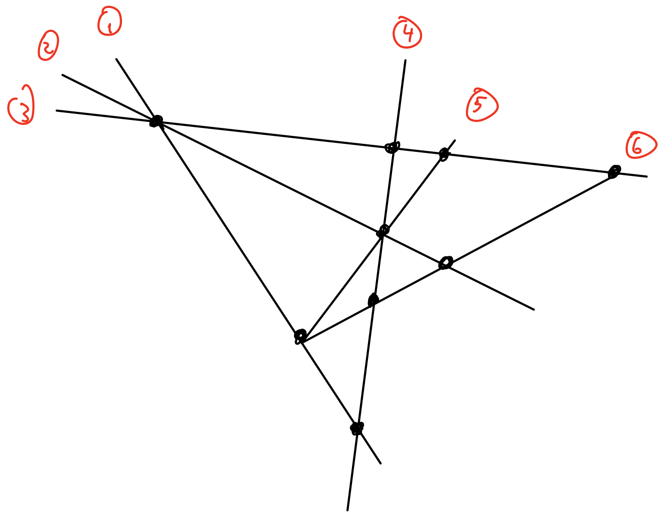
- In general, to vanish at P to multiplicity r we want the $(r-1)^{st}$ partials to vanish, so $\binom{r-1+n}{n}$ conditions. Are they independent?

- <u>Fact</u> For the complete linear system, yes.

  For a smaller linear system, not necessarily!

<u>III</u>. <u>Unexpected</u> hyper<u>surfaces</u> (connection to Lefschetz later)

<u>Example</u> (DIV) (I is in the room!)



$Z = 9$ pts. HF $\quad 1 \quad 3 \quad 6 \quad 9 \quad 9 \longrightarrow$

$$\Rightarrow \quad \dim [I_Z]_3 = 15 - 9 = 6 \quad \text{(vector space)}$$

So we get a linear system that's a $\mathbb{P}^5$.

Let P be a gen'l point. We expect $\binom{3-1+2}{2} = 6$ conditions for a curve in this linear system to vanish to order 3, ie we don't expect <u>any</u> such curve. And yet it always exists!

This example launched an active area of research on <u>unexpected</u> <u>hypersurfaces</u>.

(In this case unexpected curves, but the theory extends to $\mathbb{P}^n$.)

More precisely:

an element $F \in [I_Z]_d$ vanishes to order r at P

iff $F \in [I_Z \cap I_P^r]_d$.

Def Given a set $Z \subset \mathbb{P}^n$, positive integers d, r, and a general point P:

the actual dimension is $\text{a-dim} = \dim [I_Z \cap I_P^r]_d$.

the virtual dimension is $\text{v-dim} = \dim [I_Z]_d - \binom{r-1+n}{n}$

the expected dimension is $\text{e-dim} = \max\{\text{v-dim}, 0\}$

Z admits an unexpected hypersurface of degree d and mult. r if

$\text{a-dim} > \text{e-dim}$.

This happens in the preceeding example.

IV. (Unexpected) Cones

Def In the previous definition, if $d = r$ then Z admits an unexpected cone of degree d.

Why? Suppose S is such a hypersurface. A general line thru P already meets S in a 0-dim'l scheme of degree d so it it contains one more point of S it must lie inside S. So S is a cone.

Important Observation

Given such a cone (unexpected or not), it shows that for the projection π_P from P to a general hyperplane H, we have $\pi_P(Z)$ lies in a degree d hypersurface of H.

Example Classical Weddle Surfaces

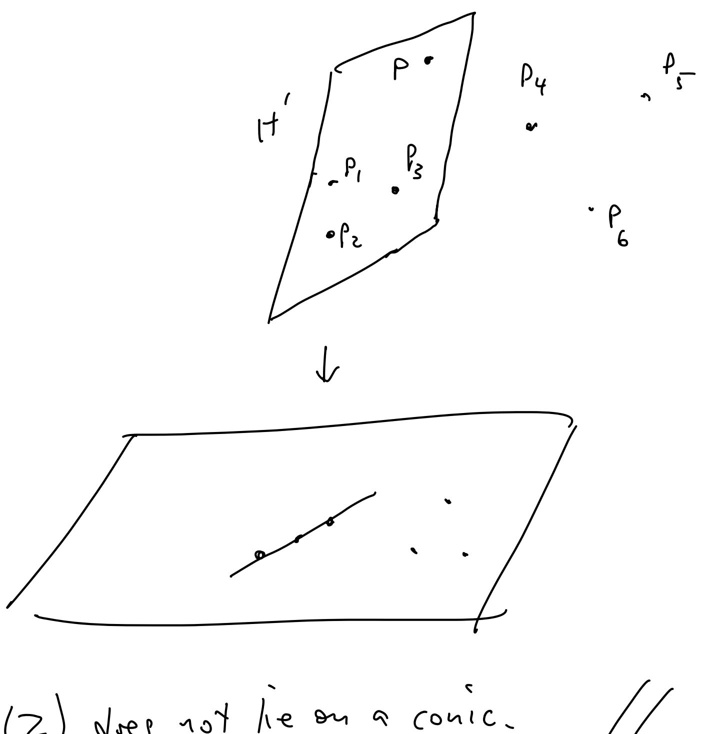
let $Z \subset \mathbb{P}^3$ be a set of 6 points in LGP. (In this case this means no four on a plane.) Note $HF = (1, 4, 6, 6, \ldots)$

Claim a general projection of Z to a hyperplane $H = \mathbb{P}^2$ does not lie on a conic curve.

proof:

let $Z = \{P_1, \ldots, P_6\}$

- It suffices to find one such projection.
- Choose any 3 of the points, say P_1, P_2, P_3, and let H' be the plane they span.
- No other point of Z lies on H'.
- let P be a general point of H' and π_P the projection from P.
- $\pi_P(P_1, P_2, P_3)$ lie on a line but $\pi_P(P_4, P_5, P_6)$ don't



$\therefore$ $\pi_P(Z)$ does not lie on a conic. //

Question: which points $P \in \mathbb{P}^3$ do have the property that $\pi_P(Z)$ lies on a conic?

Def: Let W be the set of all such points. W is the Weddle surface of Z.

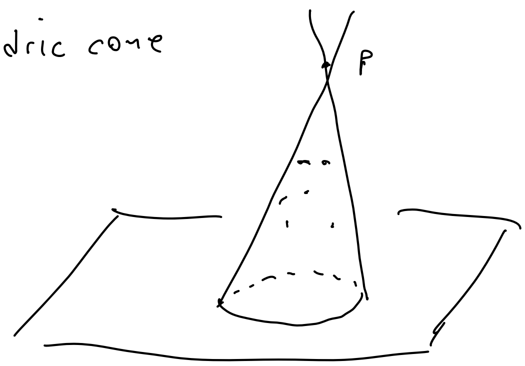
Algebraic Observation

TFAE

(1) $P \in W$

(2) $\pi_P(Z)$ lies on a conic

(3) Z lies on a quadric cone



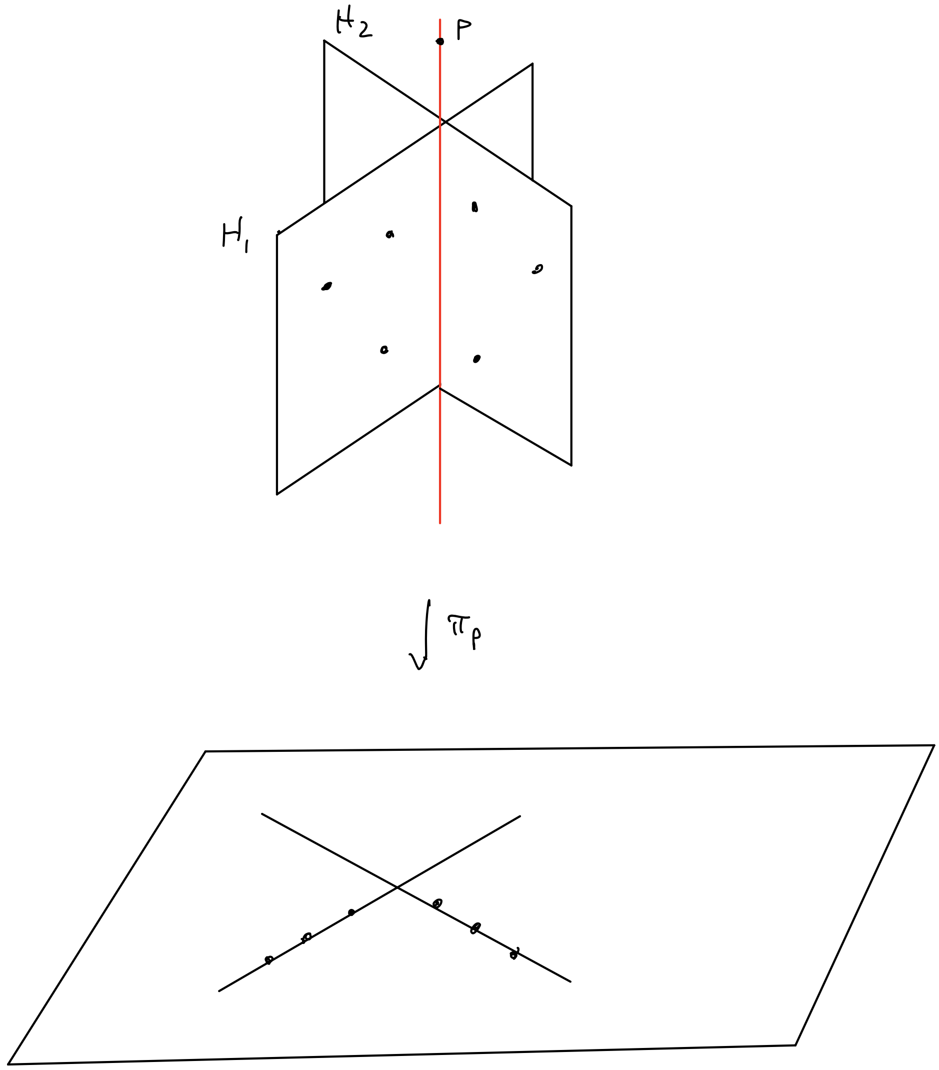
(4) $[I_Z \cap I_P^2]_2 \neq 0$

What does W look like?

First: some "obvious" subsets of W.

1. Let $\overline{P_iP_j}$ be the line joining P_i, P_j. Then projecting from any point of $\overline{P_iP_j}$ collapses P_i, P_j so $\pi_P(Z)$ has just 5 points, hence lies on a conic. So we get $\binom{6}{2} = 15$ lines on W.

2. Partition Z into 2 sets of 3 pts, e.g. $\{P_1, P_2, P_3\} \cup \{P_4, P_5, P_6\}$. This gives 2 planes H_1, H_2. Let $\lambda = H_1 \cap H_2$. Then projecting from any point of λ sends Z to 6 pts on 2 lines.



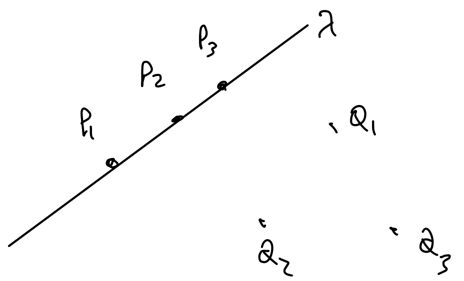
3. The points of Z lie on $\mathcal{W}$. (Special case of # 1.)

<u>Fact 1</u> $\mathcal{W}$ is actually a surface of degree 4 in $\mathbb{P}^3$. (See why next lecture.)

<u>Fact 2</u> $\mathcal{W}$ is also the non-Lefschetz locus of a certain artinian graded algebra! See why next time.

<u>Fact 3</u> If you weaken the LGP assumption a little bit you still get $\mathcal{W}$ to be a quartic surface, but not as nice as above.

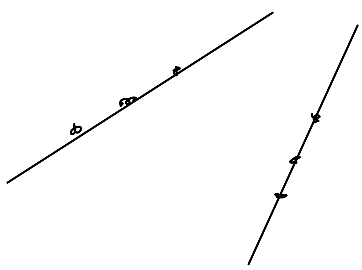
e.g. Z :



$\Rightarrow W$ is a union of 4 planes:

$$\overline{Q_1 Q_2 Q_3}, \quad \overline{\lambda Q_1}, \quad \overline{\lambda Q_2}, \quad \overline{\lambda Q_3}$$

Fact 4 More extreme:



W is all of $\mathbb{P}^3$. We'll revisit this next lecture too.

This will translate to an algebra that fails WLP. (next time)

Fact 5 There is a more general notion of Weddle loci and Weddle schemes (even in higher $\mathbb{P}^n$). See notes.

Fact 6 the example from Fact 4 has a very special property. A general projection of Z is a CI $(2,3)$. (Obvious.)

Sets of points with this property are said to be geproci (- - -)

Recently a lot of work has gone into the study of geproci sets in $\mathbb{P}^3$, and certain generalizations. We won't get into that here.

See POLITUS1. (Chiantini - Farnik - Favacchio - Harbourne - M - Szemberg - Szpond)