# Lefschetz properties of some level algebras arising from graphs

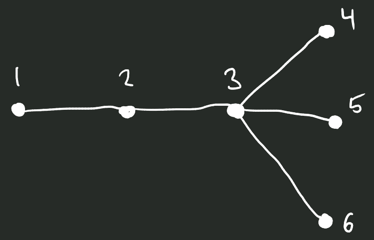
Joint work with Susan Cooper, Sara Faridi, Thiago Holleben, and Adam Van Tuyl

Lisa Nicklasson

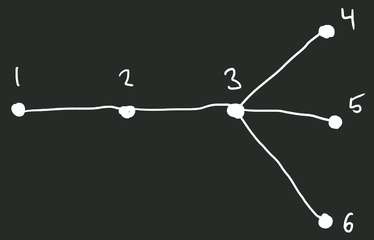
Mälardalens universitet

# Graphs !

# Graphs !

$G =$ 

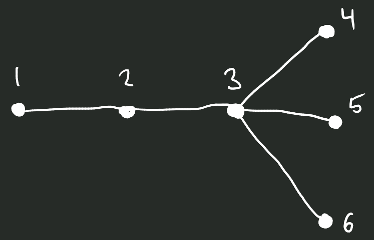
# Graphs!

$G =$ 


Independent sets:

| | |
|---|---|
| $\emptyset$ | 1 |
| $\{1\}, \{2\}, \dots, \{6\}$ | 6 |
| $\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,5\}, \{2,6\}, \{4,5\}, \{4,6\}, \{5,6\}$ | 10 |
| $\{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \{4,5,6\}$ | 7 |
| $\{1,4,5,6\}, \{2,4,5,6\}$ | 2 |

# Graphs!

$G =$ 

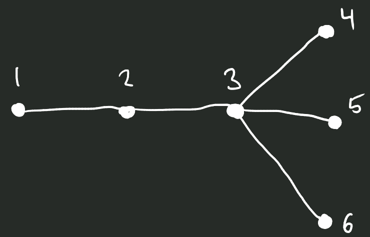


Independent sets:

| | |
|---|---|
| $\emptyset$ | 1 |
| $\{1\}, \{2\}, \ldots, \{6\}$ | 6 |
| $\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\},$ $\{2,3\}, \{2,5\}, \{2,6\},$ $\{4,5\}, \{4,6\}, \{5,6\}$ | 10 |
| $\{1,4,5\}, \{1,4,6\}, \{1,5,6\}$ $\{2,4,5\}, \{2,4,6\}, \{2,5,6\}$ $\{4,5,6\}$ | 7 |
| $\{1,4,5,6\}, \{2,4,5,6\}$ | 2 |

Independence polynomial $\quad 1 + 6t + 10t^2 + 7t^3 + 2t^4$

# Graphs!

$G =$ 

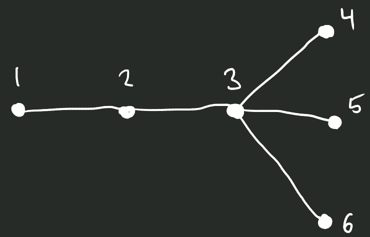

Independent sets:

| | |
|---|---|
| $\emptyset$ | 1 |
| $\{1\}, \{2\}, \ldots, \{6\}$ | 6 |
| $\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,5\}, \{2,6\}, \{4,5\}, \{4,6\}, \{5,6\}$ | 10 |
| $\{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \{4,5,6\}$ | 7 |
| $\{1,4,5,6\}, \{2,4,5,6\}$ | 2 |

maximal independent sets

Independence polynomial $1 + 6t + 10t^2 + 7t^3 + 2t^4$

# Graphs!

$G =$ 


Independent sets:

$\emptyset$ — 1

$\{1\}, \{2\}, \ldots, \{6\}$ — 6

$\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\},$
$\{2,3\}, \{2,5\}, \{2,6\},$
$\{4,5\}, \{4,6\}, \{5,6\}$ — 10

$\{1,4,5\}, \{1,4,6\}, \{1,5,6\}$
$\{2,4,5\}, \{2,4,6\}, \{2,5,6\}$
$\{4,5,6\}$ — 7

$\{1,4,5,6\}, \{2,4,5,6\}$ — 2

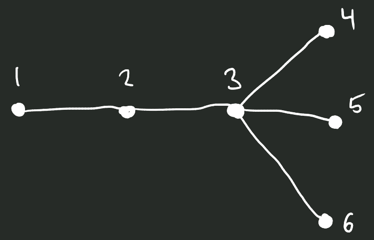
maximal independent sets

independence number

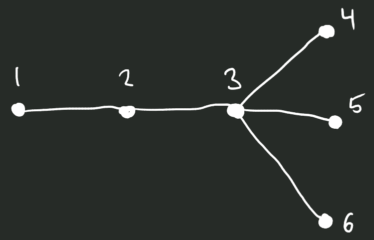
$\alpha(G) = 4$

Independence polynomial $1 + 6t + 10t^2 + 7t^3 + 2t^4$

$R = \mathbb{k}[x_1, x_2, \ldots, x_6]$

$G =$ 

$R = \Bbbk[x_1, x_2, \dots, x_6]$

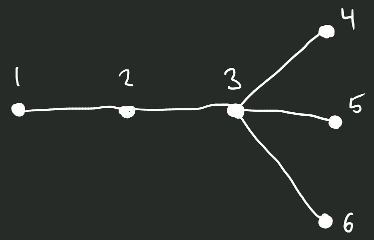
$G =$ 


Edge ideal

$I(G) = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_3x_5,\ x_3x_6)$

$R = \Bbbk[x_1, x_2, \ldots, x_6]$

$G =$




Edge ideal

$$I(G) = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_3x_5,\ x_3x_6)$$

Artinian $\Bbbk$-algebra $\quad A(G) = \dfrac{R}{I(G) + (x_1^2, \ldots, x_6^2)}$

Basis for $A(G)$

$1$

$x_1, x_2, \ldots, x_6$

$x_1x_3, \; x_1x_4, \; x_1x_5, \; x_1x_6, \ldots, x_5x_6$

$x_1x_4x_5, \; x_1x_4x_6, \; \ldots, \; x_4x_5x_6$

$x_1x_4x_5x_6, \; x_2x_4x_5x_6$

Independent sets:

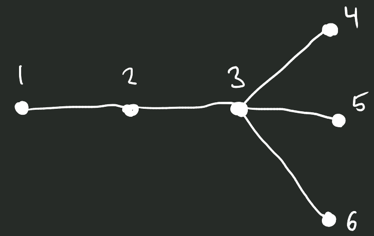
$\emptyset$

$\{1\}, \{2\}, \ldots, \{6\}$

$\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\},$
$\{2,5\}, \{2,6\}, \{4,5\}, \{4,6\}, \{5,6\}$

$\{1,4,5\}, \{1,4,6\}, \{1,5,6\}$
$\{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \{4,5,6\}$

$\{1,4,5,6\}, \{2,4,5,6\}$

$G =$ 

Basis for $A(G)$

$1$

$x_1, x_2, \ldots, x_6$

$x_1x_3, x_1x_4, x_1x_5, x_1x_6, \ldots, x_5x_6$

$x_1x_4x_5, x_1x_4x_6, \ldots, x_4x_5x_6$

$x_1x_4x_5x_6, x_2x_4x_5x_6$

Independent sets:

$\emptyset$

$\{1\}, \{2\}, \ldots, \{6\}$

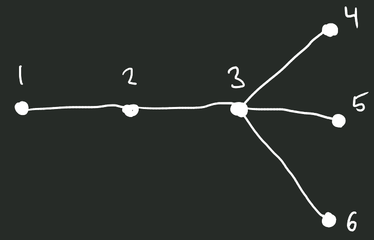
$\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\},$
$\{2,5\}, \{2,6\}, \{4,5\}, \{4,6\}, \{5,6\}$

$\{1,4,5\}, \{1,4,6\}, \{1,5,6\}$
$\{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \{4,5,6\}$

$\{1,4,5,6\}, \{2,4,5,6\}$

Hilbert series of $A(G)$ =
Independence polynomial of $G$

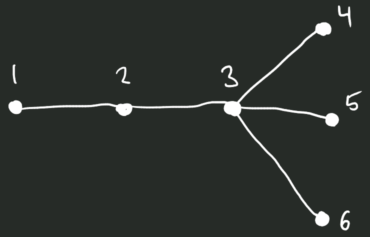
$$1 + 6t + 10t^2 + 7t^3 + 2t^4$$

$G =$ 

Socle of $A(G)$ $\longleftrightarrow$ maximal independent sets of $G$

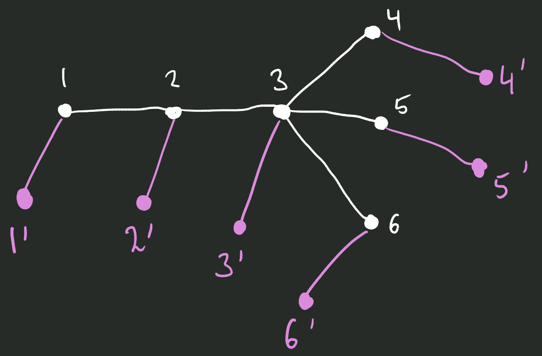
$A(G)$ is level $\longleftrightarrow$ all max. ind. sets have the same size $\alpha(G)$

# Whiskering a graph

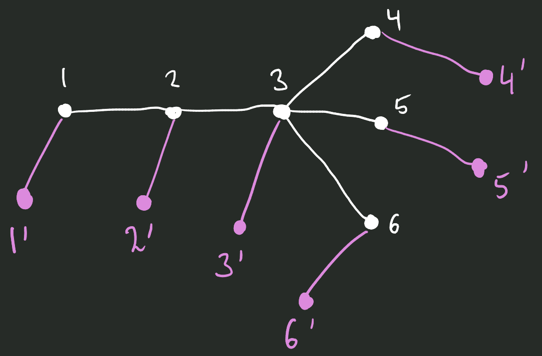
$G =$ 

# Whiskering a graph

$w(G) =$



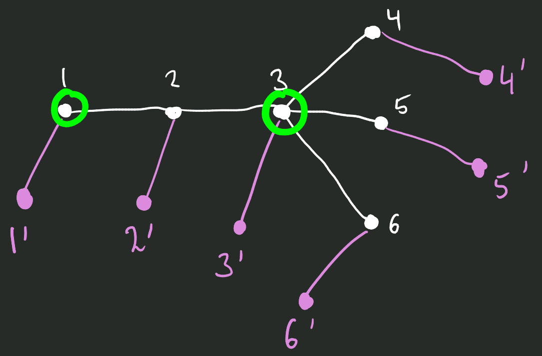
# Whiskering a graph

$w(G) =$ 


All maximal independent sets have size = #vertices in G = 6

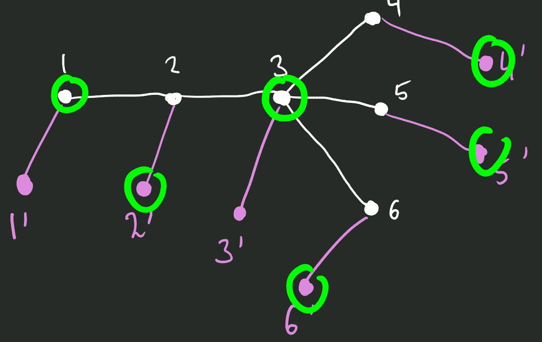
# Whiskering a graph

$w(G) =$




All maximal independent sets have size = #vertices in G = 6

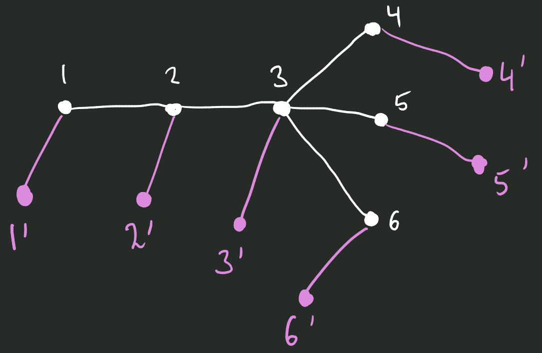
# Whiskering a graph

$w(G) =$ 


All maximal independent sets have size = #vertices in G = 6

# Whiskering a graph

$w(G) =$




$$A(w(G)) = \frac{\Bbbk[x_1, \ldots, x_6, y_1, \ldots, y_6]}{I(G) + (x_1y_1, \ldots, x_6y_6) + (x_1^2, \ldots x_6^2, y_1^2, \ldots, y_6^2)}$$

level algebra

$G =$ graph on $n$ vertices

Hilbert series of $A(w(G))$ : $h_0 + h_1 t + h_2 t^2 + \ldots + h_n t^n$

$G =$ graph on $n$ vertices

Hilbert series of $A(w(G))$ : $h_0 + h_1 t + h_2 t^2 + \dots + h_n t^n$

$$h_0 \le h_1 \le \dots \le h_{\lceil \frac{n}{2} \rceil} \qquad h_{\lceil \frac{2n-1}{3} \rceil} \ge \dots \ge h_n$$

$G =$ graph on $n$ vertices

Hilbert series of $A(w(G))$ : $h_0 + h_1 t + h_2 t^2 + \dots + h_n t^n$

$h_0 \leq h_1 \leq \dots \leq h_{\lceil \frac{n}{2} \rceil}$ $\qquad$ $h_{\lceil \frac{2n-1}{3} \rceil} \geq \dots \geq h_n$

Open problem: Is it unimodal ?

$G =$ graph on $n$ vertices

Hilbert series of $A(w(G))$ : $h_0 + h_1 t + h_2 t^2 + \dots + h_n t^n$

$h_0 \leq h_1 \leq \dots \leq h_{\lceil \frac{n}{2} \rceil}$ $\qquad$ $h_{\lceil \frac{2n-1}{3} \rceil} \geq \dots \geq h_n$

Open problem: Is it unimodal ?

WLP / SLP of $A(w(G))$ ?

# Weak & Strong Lefschetz property

$A = A_0 \oplus A_1 \oplus \cdots \oplus A_t$

has the WLP if $\exists\, \ell \in A_1$ s.t. $\cdot\, \ell : A_i \to A_{i+1}$ has max. rank $\forall i$

— SLP — " — $\cdot\, \ell^d : A_i \to A_{i+d}$ — " — $\forall i, d$

Lemma. Let $A$ be a monomial level algebra of characteristic $0$ and socle degree $t$. Then

- $\ell : A_i \to A_{i+1}$ is injective for $i < \frac{t}{2}$

Hausel 2005

**Theorem.** $G$ = graph on $n$ vertices and at least one edge
$A = A(w(G))$ char. $\neq 2$
Then
- $\ell : A_1 \to A_2$ and
- $\ell : A_{n-1} \to A_n$

have maximal rank.

Application of Dao-Nair 2024

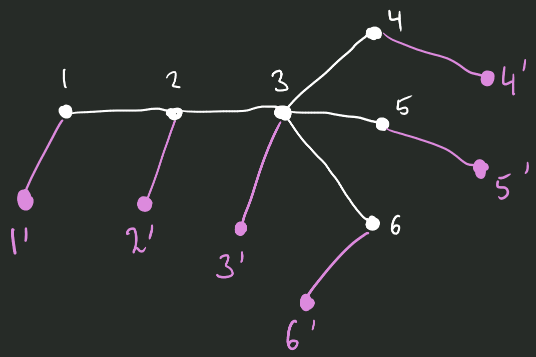
Theorem. $A(w(K_n))$ has SLP.

Theorem. $A(w(G))$ has WLP if $\alpha(G) \leq 2$, and does not have WLP if $\alpha(G) \geq \frac{n}{3} + 2$.

**Theorem.** $A(w(K_n))$ has SLP.

**Theorem.** $A(w(G))$ has WLP if $\alpha(G) \le 2$, and does not have WLP if $\alpha(G) \ge \frac{n}{3} + 2$.

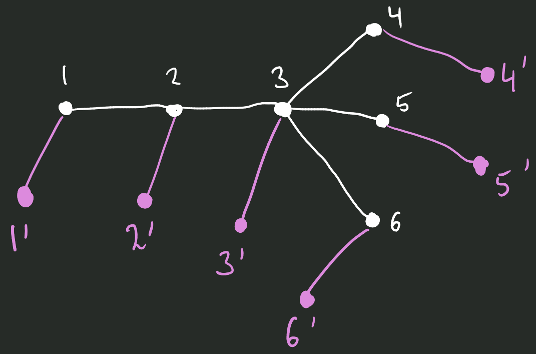
**Question:** Is WLP of $A(w(G))$ determined by $\alpha(G)$?

$w(G) =$ 


$$A(w(G)) = \frac{\mathbb{k}[x_1, \dots, x_6, y_1, \dots, y_6]}{I(w(G)) + (x_1^2, \dots, x_6^2, y_1^2, \dots, y_6^2)}$$

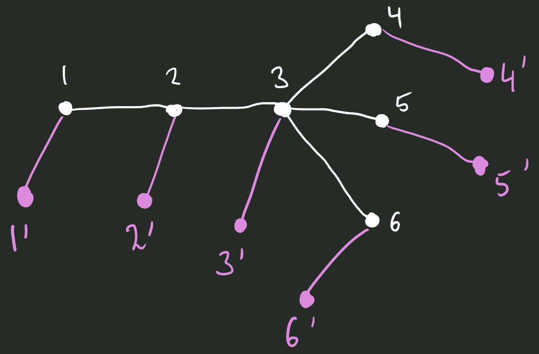
$$\alpha(G) = 4 \geq \frac{6}{3} + 2 \qquad \text{not WLP}$$

$w(G) =$




$$A(w(G)) = \frac{\mathbb{k}[x_1, \ldots, x_6, y_1, \ldots, y_6]}{I(w(G)) + (x_1^2, \ldots x_6^2, y_1^2, \ldots, y_6^2)}$$

$$1 + 12T + 55T^2 + 127T^3 + 158T^4 + 101T^5 + 26T^6$$

$w(G) =$ 


$$A(w(G)) = \frac{\mathbb{k}[x_1, \ldots, x_6, y_1, \ldots, y_6]}{I(w(G)) + (x_1^2, \ldots, x_6^2, y_1^2, \ldots, y_6^2)}$$

$$1 + 12T + 55T^2 + 127T^3 + 158T^4 + 101T^5 + 26T^6$$

$\cdot\ \ell$

# Thank you!

Reference. S. Cooper, S. Faridi, T. Holleben, L. Nicklasson, A. Van Tuyl.
*The weak Lefschetz property of whiskered graphs*
In Lefschetz Properties: Current and New Directions
or arXiv: 2306.04393