WLP for certain ideals generated by powers of linear forms

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On the Weak Lefschetz Property for certain ideals generated by powers of linear forms

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Problem

Setup

- k is a field of charachteristic 0;
- $R = k[x_0, \ldots, x_n] = k[\mathbb{P}^n]$ is a standard graded polynomial ring.

$$P = [p_0 : \ldots : p_n] \in \mathbb{P}^n \longleftrightarrow L_P = \sum p_i x_i \in [R]_1$$

$$\begin{matrix} X = \{P_1, \ldots, P_r\} \subseteq \mathbb{P}^n \\ \text{and} \\ d \in \mathbb{Z}_+ \end{matrix} \quad \longleftrightarrow \quad \Lambda_{X,d} = (L_{P_1}^d, \ldots, L_{P_r}^d) \subseteq R$$

Problem:

Study the WLP of $R/\Lambda_{X,d}$

Some known results

$X \subseteq \mathbb{P}^1$ $X \subseteq \mathbb{P}^2$	$R/\Lambda_{X,d}$ has the WLP $\forall d$ Harima, Migliore, Nagel, Watanabe (2003) Schenck, Seceleanu(2010)
X is a set of $n+1$ general points in $\mathbb{P}^n$	$\Lambda_{X,d}$ is a CI $R/\Lambda_{X,d}$ has the WLP $\forall d$ Stanley(1980)
X is a set of $n+2$ general points in $\mathbb{P}^n$	$\Lambda_{X,d}$ is an ACI $R/\Lambda_{X,d}$ has the WLP if and only if $d = 1$ or $n \leq 2$ or $\begin{array}{c\|cccc} n & 3 & 4 & 4 & 6 \\ \hline d & 2 & 2 & 3 & 2 \end{array}$ Boij, Lundqvist(2023), Harbourne,Ilardi,Migliore,Miró-Roig,Nagel, Schenck,Seceleanu,Tran,Trok,Vallès
X admits an unexpected cone of degree δ	$R/\Lambda_{X,\delta}$ fails the WLP Harbourne, Migliore, Nagel, Teitler (2021)

Study the WLP

$$X = \{P_1, \dots, P_r\} \subseteq \mathbb{P}^n,\ d \in \mathbb{Z}_+ \quad \longleftrightarrow \quad \Lambda_{X,d} = (L_{P_1}^d, \dots, L_{P_r}^d)$$

$$P \text{ general point in } \ \mathbb{P}^n \quad \longleftrightarrow \quad L_P$$

$$\left[R/\Lambda_{X,d}\right]_{d+t-1} \xrightarrow{\times L_P} \left[R/\Lambda_{X,d}\right]_{d+t} \longrightarrow \left[R/(L_{P_1}^d, \dots, L_{P_r}^d, L_P)\right]_{d+t} \longrightarrow 0$$

for $t \geq 0$

$$WLP \longleftrightarrow \dim\left[R/(L_{P_1}^d, \dots, L_{P_r}^d, L_P)\right]_{d+t} = \max\left\{0, \Delta h_{R/\Lambda_{X,d}}(d+t)\right\}$$

Macaulay duality. Emsalem-Iarrobino

$X = \{P_1, \dots, P_r\} \subseteq \mathbb{P}^n$, P general point, $\Lambda_{X,d} = (L_{P_1}^d, \dots, L_{P_r}^d)$

$$[R/\Lambda_{X,d}]_{d+t} \cong [I_{P_1}^{t+1} \cap \cdots \cap I_{P_r}^{t+1}]_{d+t} = [I_X^{(t+1)}]_{d+t}$$

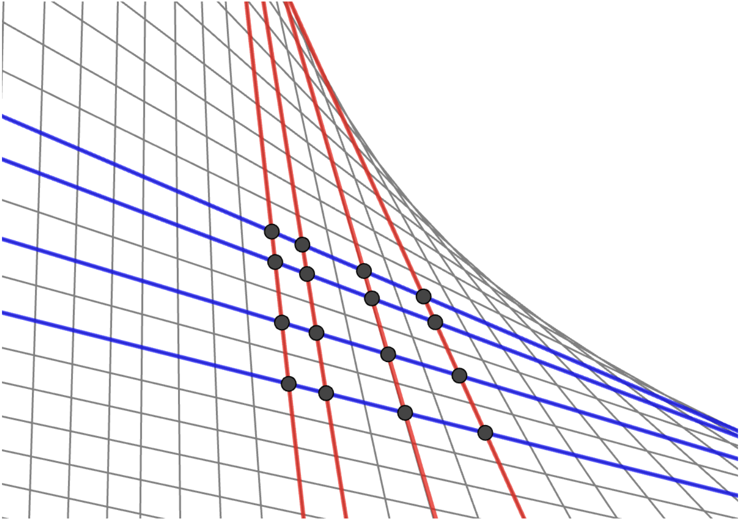
$$\Delta h_{R/\Lambda_{X,d}}(d+t) = \dim[I_X^{(t+1)}]_{d+t} - \dim[I_X^{(t)}]_{d+t-1}$$

$$\begin{aligned} \dim[R/(L_{P_1}^d, \dots, L_{P_r}^d, L_P)]_{d+t} &= \dim[I_{P_1}^{t+1} \cap \cdots \cap I_{P_r}^{t+1} \cap I_P^{d+t}]_{d+t} \\ &= \dim[I_X^{(t+1)} \cap I_P^{d+t}]_{d+t} \end{aligned}$$

$$\dim[\mathrm{Coker}(\times L_P)]_{d+t} = \dim \left[I_{\pi_P(X)}^{(t+1)} \right]_{d+t}$$

Migliore, Miró-Roig, Nagel (2012)

$a \times a$-grids in $\mathbb{P}^3$



$a \times a$-grid, $d = a$

$a \times a$-grids, $a \geq 3$, X admit unexpected cones of degree a
Chiantini, Migliore (2021)

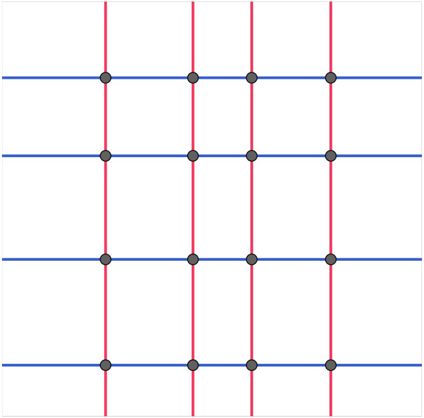
Then from HMNT we get

$R/\Lambda_{X,a}$ fails the WLP.

Chiantini, Farnik, F–, Harbourne, Migliore, Szemberg, Szpond(2022)

$a \times a$-grid, $d \geq a - 1$

$$0 \to I_X^{(t)}(-2) \xrightarrow{\times Q} I_X^{(t+1)} \to \frac{I_X^{(t+1)} + (Q)}{(Q)} \to 0$$



$$h_{(t+1)X|_Q}(i) = h_{(t+1)Z}(i,i) \text{ where } Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$$

$a \times a$-grid, $\Lambda_{X,d}$, $d \geq a-1$

$a \times a$-grid X is (a,a)-geproci set in $\mathbb{P}^3$

$$\pi_P(X) \text{ is a CI} \longrightarrow I^{(t)}_{\pi_P(X)} = I^t_{\pi_P(X)} \quad \forall t \geq 0$$

$d = (a-1)\,\mathrm{q} + \mathrm{r} \qquad 0 \leq r < a-1$ and $q \geq 1$

$$\times L_P : [R/\Lambda_{X,d}]_{d+q-2} \to [R/\Lambda_{X,d}]_{d+q-1}$$

$$\Delta h_{\Lambda_{X,d}}(d+q-1) < \dim coker(\times L_P) = (q+1)\binom{r+1}{2}$$

$$r \neq 0 \longrightarrow R/\Lambda_{X,d} \text{ fails the WLP}$$

WLP for $a \times a$-grids

$d = (a-1)q + r \quad 0 \le r < a-1$ and $q \ge 1$

if $r = 0$ then

$$\Delta h_{\Lambda_{X,d}}(d+q-1) < \dim coker(\times L_P) = 0$$

$$\Delta h_{R/\Lambda_{X,d}}(d+q-2) = q\binom{a}{2}$$

$$\Delta h_{R/\Lambda_{X,d}}(d+q-1) = -q\binom{a}{2}$$

$r = 0 \longrightarrow R/\Lambda_{X,d}$ has the WLP!

$a \times a$-grid, $d \leq a-1$

$a \times a$-grid X on a smooth quadric $\mathcal{Q} = \mathbb{V}(Q)$
$d \leq a-1$

$$\begin{aligned} \Lambda_{X,d} = & \quad (L_P^d \mid P \in X) \\ = & \quad (L_P^d \mid P \in \mathcal{Q}) \\ = & \quad (L_P^d \mid P \in \text{ any } (d+1) \times (d+1) - \text{grid in } \mathcal{Q}) \\ = & \quad (Q^{d-1})^{\perp} \end{aligned}$$

$R/\Lambda_{X,d}$ is a compressed Gorenstein algebra of even socle degree.

$R/\Lambda_{X,d}$ has the WLP

Behaviour of the WLP

$a \times a$ grid X in $\mathbb{P}^3$ $R/\Lambda_{X,d}$

d	1	2	$\cdots$	$a-1$	a	$\cdots$	$2(a-1)-1$	$2(a-1)$	$\cdots$
WLP	Y	Y	$\cdots$	Y	N	$\cdots$	N	Y	$\cdots$
	1	1	$\cdots$	1	0	$\cdots$	0	1	$\cdots$

$$b_X = 0.b_1 b_2 \ldots \in \mathbb{R} \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } R/\Lambda_{X,i} \text{ has WLP} \\ 0 & \text{if } R/\Lambda_{X,i} \text{ fails WLP} \end{cases}$$

$$b_X = 0.\underbrace{1\ldots1}_{a-2}\underbrace{\overline{10\ldots0}}_{a-1} \in \mathbb{Q}$$

Questions:

1. Let X be a set of points in $\mathbb{P}^n$. Is always $b_X \in \mathbb{Q}$?
2. Let $b = 0.b_1 b_2 \ldots \in \mathbb{R}$ be such that $b_i \in \{0, 1\}$. When $b = b_X$, for X set of points in $\mathbb{P}^n$?

Thank you!