Commuting Jordan Types

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Setup

k an infinite field

$\mathcal{N}ilp_n(\mathrm{k})$: $n \times n$ nilpotent matrices with entries in k.

For $A \in \mathcal{N}ilp_n(\mathrm{k})$, the nilpotent commutator of A is defined as

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For $A \in \mathcal{N}ilp_n(\mathrm{k})$, the unique partition of n given by the size of Jordan blocks in the Jordan canonical form of A is called the *Jordan type* of A and is denoted by P_A.

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Definition

Let P and Q be partitions of n. We say P and Q *commute* if there exist $A, B \in \mathcal{N}ilp_n(\mathrm{k})$ such that $AB = BA$, $P_A = P$ and $P_B = Q$.

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Lemma

Let $B \in \mathcal{N}ilp_n(\mathsf{k})$. For $i \geq 1$, let $d_i = \operatorname{rank} B^{i-1} - \operatorname{rank} B^i$. Then the partition $d = (d_1, d_2, \dots)$ and P_B are conjugate partitions.

Jordan types commuting with (4)

$$B = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B^2 = \begin{pmatrix} 0 & 0 & a^2 & 2ab \\ 0 & 0 & 0 & a^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B^3 = \begin{pmatrix} 0 & 0 & 0 & a^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B^4 = 0$$

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	$(\operatorname{rank} B^i)_{i=0,1,\ldots}$	d	P_B
General B	$(4,3,2,1,0)$	$(1,1,1,1)$	(4)
$a=0,\ b\neq 0$	$(4,2,0)$	$(2,2)$	$(2,2)$
$a=b=0,\ c\neq 0$	$(4,1,0)$	$(3,1)$	$(2,1,1)$
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Jordan type (4) commutes with all partitions of 4, except for $(3,1)$.

Proposition (Basili, 2003)

$\mathcal{N}_A$ is an irreducible algebraic variety.

Definition

Let $\mathcal{P}(n)$ be the set of all partitions of n. Let $\mathcal{Q} : \mathcal{P}(n) \to \mathcal{P}(n)$ be the map that sends a partition P to $\mathcal{Q}(P)$, the generic Jordan type that commutes with P.

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Goal

Explicitly determine $\mathcal{Q}$

Example

Recall that every $P \in \mathcal{P}(4) - \{(3,1)\}$ commutes with (4). Moreover, (4) dominates all partitions of 4. Therefore,

$$\mathcal{Q}(P) = \begin{cases} (4) & \text{if } P \neq (3,1) \\ (3,1) & \text{if } P = (3,1). \end{cases}$$

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- If $\dim_{\mathsf{k}} \mathsf{k}[A, B] = n$, then the Jordan type of the general linear form in $\mathsf{k}[A, B]$, as well as its associated graded algebra, is conjugate to the partition obtained by the Hilbert function of $\mathsf{k}[A, B]$. $\rightsquigarrow\rightsquigarrow$ SLP

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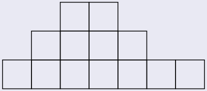
- $\mathcal{Q}(P) = P$ if and only if parts of P differ pairwise by at least 2.

Theorem (Košir and Oblak, 2009)

If B is generic in $\mathcal{N}_A$, then $\mathsf{k}[A, B]$ is Gorenstein.

Macaulay: Hilbert function of a height 2 Artinian Gorenstein (CI) algebra $(1, 2, \ldots, d, h_d, h_{d+1}, \ldots, h_{j-1}, 1)$, where $h_i - h_{i+1} \leq 1$ for $d \leq i \leq j$.



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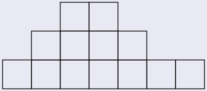

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Corollary

The map $\mathcal{Q}$ is idempotent.

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Example

For $P = (5,3,2^2,1^2)$, $r_P = 3$. Indeed,
$P = (\underline{5}, \underline{3, 2^2}, \underline{1^2}) = (\underline{5}, \underline{3}, \underline{2^2, 1^2})$.

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Consider $P = (5, 3, 2^2, 1^2)$. Note that each almost rectangular subpartition commutes with a regular partition of the appropriate size.

So P commutes with $(7, 5, 2)$, as well as $(6, 5, 3)$. Since $(6, 5, 3) < (7, 5, 2)$, not to mention that $(6, 5, 3)$ can't be in the image of $\mathcal{Q}$, maybe $\mathcal{Q}(P) = (7, 5, 2)$??

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NO! In fact, $\mathcal{Q}(P) = (9, 4, 1)$.

Oblak Process

- Let P be a partition of n and write it as $P = (p_1^{n_1}, p_2^{n_2}, \ldots, p_t^{n_t})$, where $p_1 > \cdots > p_t$ and $n_i > 0$, for all i.

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For each almost rectangular subpartition of P, take all vertices in rows of $\mathcal{D}_P$ corresponding to the almost rectangular subpartition of P, as well as the first and the last vertex in any row "above" it. This is a chain in $\mathcal{D}_P$ and is called a U-chain.

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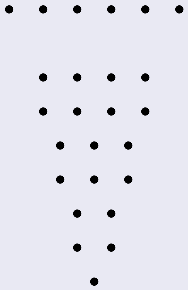
$P = (6, 4^2, 3^2, 2^2, 1)$



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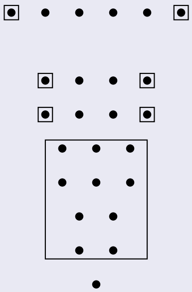

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- Find a U-chain in $\mathcal{D}_P$ with maximum possible cardinality. Remove the chain to obtain a new partition, and repeat.
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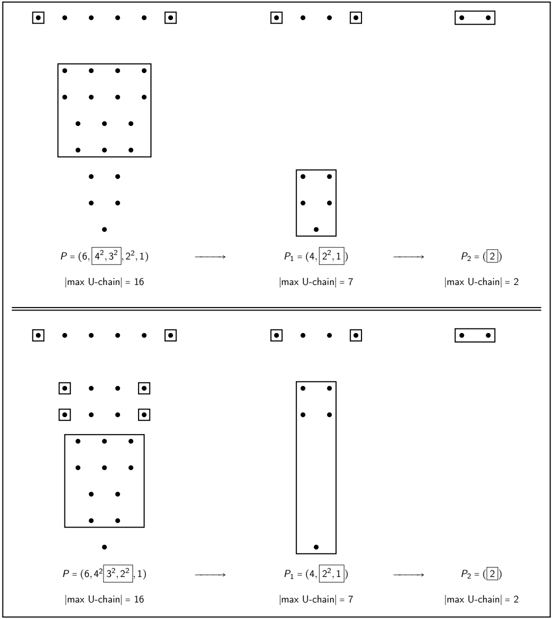
- Oblak's conjecture: $\mathrm{Obl}(P) = \mathcal{Q}(P)$.



$$\mathcal{Q}(6, 4^2, 3^2, 2^2, 1) = (16, 7, 2)$$

Theorem (K, 2013)

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More about $\mathcal{Q}(P)$

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Theorem (Basili, 2022)

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Box Conjecture

Let $Q = (q_1, \ldots, q_k)$ where $q_i - q_{i+1} \geq 2$, for $1 \leq i < k$. Set

$$\delta_i = \begin{cases} q_i - q_{i+1} - 1, & \text{if } 1 \leq i < k, \\ q_k, & \text{if } i = k. \end{cases}$$

Then $\mathcal{Q}^{-1}(Q)$ consists of $\prod \delta_i$ partitions which can be arranged in a $\delta_1 \times \cdots \times \delta_k$ "box", such that $P_{i_1,\ldots,i_k}$ has $\sum_j i_j$ parts.

In a recent preprint, Irving, Košir, Mastnak, prove the box conjecture in its general form using Shayman's results on invariant subspaces, and Burge's encoding of partitions as binary words.

Using these tools, they also give a new independent proof for Oblak's conjecture about $\mathcal{Q}(P)$.

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- Let A be an Artinian algebra (graded or local). Study Jordan types of multiplication maps $\times f : A \to A$ for f in the maximal ideal.

Dziękuję!

For $Q = (q_1, \ldots, q_k)$, there is a 1-1 correspondence between partitions P with $\mathcal{Q}(P) = Q$ and length $q_1 + 1$ binary words in $\{\alpha, \beta\}$, such that the only $\beta\alpha$ drops happen in letters $q_k, \ldots, q_1$.

$$\begin{array}{ccccc} & q_k & & q_{k-1} & & q_1 \\ & \wr & & \wr & & \wr \\ \ldots & \beta\ \alpha & \ldots & \beta\ \alpha & \ldots\ldots & \beta\ \alpha \end{array}$$

Example

$\mathcal{Q}^{-1}(7,3)$

$(7,3)$	$(7,2,1)$	$(7,1^3)$
$(5,3,2)$	$(4,3,2,1)$	$(4,3,1^3)$
$(5,2^2,1)$	$(5,2,1^3)$	$(5,1^5)$

$\alpha\alpha\beta\alpha\alpha\alpha\beta\alpha$	$\alpha\beta\beta\alpha\alpha\alpha\beta\alpha$	$\beta\beta\beta\alpha\alpha\alpha\beta\alpha$
$\alpha\alpha\beta\alpha\alpha\beta\beta\alpha$	$\alpha\beta\beta\alpha\alpha\beta\beta\alpha$	$\beta\beta\beta\alpha\alpha\beta\beta\alpha$
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