The Strong Lefschetz Property for Modules over Clements-Lindström Rings

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Setting

- Let k be a field of characteristic zero.
- $S = k[x_1, \dots, x_n]$ standard graded polynomial ring
- $A = S/I = \bigoplus_{i=0}^{c} A_i$ standard graded Artinian k-algebra
- $M = \bigoplus_{i=a}^{b} M_i$ finitely generated A-module
- We denote the Hilbert function of M by $i \mapsto h_i := \dim_k(M)$ and the Hilbert series by $H_M(t)$

The Lefschetz Properties (for Modules)

Definition

M has the **weak Lefschetz property** (WLP) if there exists a linear form $\ell \in A_1$ such that the multiplication map

$$\times \ell \colon M_i \to M_{i+1}$$

has maximal rank for all i.

Definition

M has the **strong Lefschetz property** (SLP) if there exists a linear form $\ell \in A_1$ such that the multiplication map

$$\times \ell^d \colon M_i \to M_{i+d}$$

has maximal rank for all $d > 0$ and all i.

Modules over Rings with the WLP or SLP

- Much less is known about the Lefschetz properties for modules than for algebras. Are there interesting classes of modules which do or do not have the WLP or SLP?
- If $A = S/I$ is an Artinian k-algebra which has the WLP or SLP, which A-modules have the WLP or SLP?
- The first case we consider for modules is that of ideals in Clements-Lindström rings in two variables. A **Clements-Lindström** ring has of the form $S/(x_1^{a_1}, \ldots, x_n^{a_n})$ with $a_1 \leq \cdots \leq a_n \leq \infty$ (where $x_i^{\infty} = 0$). In this context, we may assume all Clements-Lindström rings are Artinian, i.e., complete intersections.
- **Question:** Given a monomial ideal $I \subset S/J$, where $S = k[x, y]$ and $J = (x^a, y^b)$, does I have the SLP as an S/J-module? Equivalently, does I/J have the SLP as an S-module?

The Problem With Modules

Even in codimension two, where any k-algebra A is guaranteed to have the SLP, the WLP can fail for A-modules which are monomial ideals in A.

Example

Let $J = (x^5, y^5)$ be an ideal in $S = k[x, y]$. Consider the generic initial ideal in degree reverse lexicographic order,
$\text{gin}(J) = \text{Lex}(J) = (x^5, x^4y, x^3y^3, x^2y^5, xy^7, y^9)$. Then $S/\text{gin}(J)$ has the SLP.
However, every monomial ideal I which does not contain x or y fails the SLP as an $S/\text{gin}(J)$-module.
On the other hand, I does have the SLP as an S/J module (we prove this later).

Additionally, tools used to prove the WLP/SLP for k-algebras can fail when we attempt to use them for modules (or just don't apply).

The Lemma that Saves the Day

The **Lindström-Gessel-Viennot Lemma** gives a combinatorial interpretation for the determinant of a matrix of binomial coefficients as a count of non-intersecting lattice paths.

In two variables, the matrix representing the map of multiplication by $\times(x+y)$ on any $k[x,y]$-module is a matrix of binomial coefficients.

Corollary (Gessel-Viennot, 1985)

Let $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$ be two sets of integers. Then the determinant of the matrix $C = (c_{ij})$ of binomial coefficients $c_{ij} = \binom{a_i}{b_j}$ is nonzero if and only if $\binom{a_i}{b_i} \neq 0$ for each i.

In Which We Put the Lemma to Good Use

Theorem

Let I be a monomial ideal in $S = k[x, y]$ and $J = (x^a, y^b)$ an Artinian ideal in S such that $J \subset I$. Then I/J has the SLP as a $k[x, y]$-module

The proof relies on the LGV Lemma. As an immediate corollary, we get the useful result:

Corollary

Let $I = (x^\alpha, y^\beta)$ and $J = (x^a, y^b)$ be ideals in S where $0 \leq \alpha \leq a$ and $0 \leq \beta \leq b$. Then I/J has the SLP as a $k[x, y]$-module.

In Which We Put the Lemma to Good Use

Sketch of proof: Matrix representing the multiplication map $\times(x+y)^d \colon S_i \to S_{i+d}$ (transposed):

$$\begin{array}{c} \\ x^i \\ x^{i-1}y \\ \vdots \\ y^i \end{array} \begin{array}{c} \begin{array}{cccccccc} x^{i+d} & x^{i+d-1}y & x^{i+d-2}y^2 & \dots & x^iy^d & x^{i-1}y^{d+1} & \dots & y^{i+d} \end{array} \\ \begin{bmatrix} \binom{d}{0} & \binom{d}{1} & \binom{d}{2} & \dots & \binom{d}{d} & 0 & \dots & 0 \\ 0 & \binom{d}{0} & \binom{d}{1} & \dots & \binom{d}{d-1} & \binom{d}{d} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \binom{d}{d-i} & \binom{d}{d-i+1} & \dots & \binom{d}{d} \end{bmatrix} \end{array}$$

Deleting columns corresponding to monomials in J and rows corresponding to monomials not in I gives a matrix representing $\times(x+y)^d \colon [I/J]_i \to [I/J]_{i+d}$ where the nth row is of the form

$$\begin{bmatrix} \binom{d}{k_n} & \binom{d}{k_n+1} & \dots & \binom{d}{k_n+c-1} \end{bmatrix}.$$

In Which We Put the Lemma to Good Use

Then doing a series of iterative column operations relying on the recursive relation of binomial coefficients, we finally get a matrix of the form

$$\begin{bmatrix} \binom{d+c-1}{k_1+c-1} & \binom{d+c-2}{k_1+c-1} & \cdots & \binom{d}{k_1+c-1} \\ \binom{d+c-1}{k_2+c-2} & \binom{d+c-2}{k_2+c-2} & \cdots & \binom{d}{k_2+c-2} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{d+c-1}{k_s+c-1} & \binom{d+c-2}{k_s+c-1} & \cdots & \binom{d}{k_s+c-1} \end{bmatrix}.$$

Based on our hypotheses on I and J, the fact that S and S/J have the SLP and that S/J is Gorenstein, we can prove that at least the binomial coefficients on the main diagonal of a particular maximal minor of are nonzero.

Applying the corollary of the LGV Lemma, this implies maximal rank of the original matrix for I/J, hence I/J has the strong Lefschetz property.

Modules over k[x,y,z]

It is not necessarily true that similar modules in three variables have the SLP. For example, the $k[x, y, z]$-module I/J where $I = (x^2, y^2, z^2)$ and $J = (x^3, y^3, z^3)$ fails the WLP and SLP.

However, we can consider tensor product extensions using the following:

Theorem (Lindsey, 2011)

For $M = \bigoplus_{j=p}^{q} M_i$ a graded Artinian S-module with the SLP, $M \otimes_k k[z]/(z^c)$ has the SLP for all $c \geq 0$ if and only if the Hilbert series of M is almost centered, i.e., $h_{p+i-1} \leq h_{q-i} \leq h_{p+i}$ for all $1 \leq i \leq \lfloor \frac{q-p}{2} \rfloor$ or $h_{q-i+1} \leq h_{p+i} \leq h_{q-i}$ for all $1 \leq i \leq \lfloor \frac{q-p}{2} \rfloor$.

Theorem

Let $M = (x^\alpha, y^\beta)/(x^a, y^b, z^c)$ be a $k[x, y, z]$-module where $0 \leq \alpha \leq a$ and $0 \leq \beta \leq b$. Then M has the SLP for all $c \geq 0$ if $\alpha \neq \beta$ and $\max\{\alpha, \beta\} \leq 2$ or $\max\{\alpha, \beta\} = \min\{\alpha + \beta, a, b\}$.

Central Simple Modules

- Let A be a standard graded Artinian k-algebra, $\ell \in A$ a linear form with r the smallest positive integer for which $\ell^p = 0$. Harima and Watanabe defined the **ith central simple module** of A with respect to ℓ as the nonzero quotients of the form
$$V_{i,\ell} = \frac{(0 : \ell^{f_i}) + (\ell)}{(0 : \ell^{f_i - 1}) + (\ell)}.$$
where $r \geq f_i > f_{i+1} \geq 1$ for all i.
- Harima and Watanabe (2007) gave an equivalent condition for a graded Artinian k-algebra with a symmetric Hilbert series having the SLP in terms of central simple modules. A weaker version is true even for non-symmetric Hilbert series:

Theorem

Let A be a graded Artinian k-algebra with a Hilbert series that is not necessarily symmetric. Then A has the SLP if there exists a linear form $\ell \in A_1$ such that $\widetilde{V}_\ell = \oplus(V_{i,\ell} \otimes k[t]/(t^{f_i}))$ has the SLP.

Codimension 3 k-algebras of type 2

- Boij, Migliore, Miró-Roig, Nagel, and Zanello proved that level monomial algebras of type two in codimension 3 have the WLP. In the non-level case, Cook and Nagel completely characterized when such algebras as

$$A = k[x, y, z]/(x^a, y^b, z^c, x^\alpha z^\gamma, y^\beta z^\gamma)$$

  have the WLP. Their combinatorial method of proof involved lattice paths, perfect matchings, and lozenge tilings of triangular regions. But this method cannot also be used to study the SLP.
- However, the central simple modules of A (with respect to z) are of the form

$$V_{1,z} = \frac{(0 : z^c) + (z)}{(0 : z^{c-1}) + (z)} \cong \frac{k[x, y]}{(x^\alpha, y^\beta)}$$
$$V_{2,z} = \frac{(0 : z^\gamma) + (z)}{(0 : z^{\gamma-1}) + (z)} \cong \frac{(x^\alpha, y^\beta)}{(x^a, y^b)}.$$

Notice that $V_{2,z}$ has the SLP by the previous corollary (and $V_{1,z}$ has the SLP since it is a monomial complete intersection)!

Codimension 3 k-algebras of type 2

Tensoring $V_{1,z}$ with $k[z]/(z^c)$ gives another monomial complete intersection with the SLP, and

$$V_{2,z} \otimes k[z]/(z^\delta) \cong \frac{(x^\alpha, y^\beta)}{(x^a, y^b, z^\gamma)}$$

which we saw (using Lindsey's result) also has the SLP in some cases. By looking at the Hilbert series of the above modules, we can study when the direct sum $\widetilde{V}_\ell$, and hence A, has the SLP:

Theorem

Let $I = (x^a, y^b, z^c, x^\alpha z^\gamma, y^\beta z^\gamma)$ be an Artinian monomial ideal in $S = k[x, y, z]$ where $0 < \alpha < a$, $0 < \beta < b$, and $0 < \gamma < c$. Then S/I has the strong Lefschetz property if any of the following conditions hold:

1. $\alpha + \beta - 1 \leq a + b - c \leq \alpha + \beta + 1$;
2. $min\{\alpha, \beta\} \neq max\{\alpha, \beta\} = min\{\alpha + \beta, a, b\}$ *and* $max\{\alpha, \beta\} - \gamma - 1 \leq a + b - c \leq max\{\alpha, \beta\} - \gamma + 1$;
3. $min\{\alpha, \beta\} < max\{\alpha, \beta\} \leq 2$ *and* $a + b + \gamma \leq c + 2$.

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