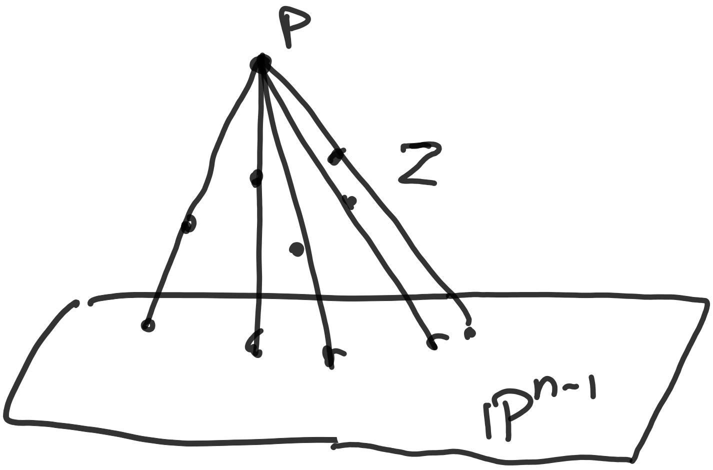
LEFSCHETZ PROPERTIES IN ALGEBRA, GEOMETRY, TOPOLOGY AND COMBINATORICS CONFERENCE, KRAKÓW, JUNE 23-29, 2024

PROJECTIONS and WEDDLE LOCI.

joint with
- POLITUS GROUP
- F Fagioli

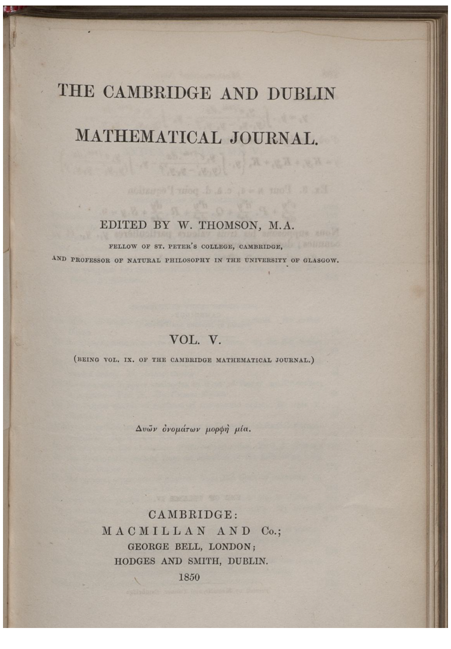
$\mathbb{P}^n_{\mathbb{C}}$



$\mathcal{P}$ = property of sets in $\mathbb{P}^{n-1}$

$W_{\mathcal{P}}(Z) = \{ P \in \mathbb{P}^n :$ the set-theoretic projection π_P sends Z to a set $\pi_P(Z)$ which enjoys $\mathcal{P} \}$.

HISTORICAL NOTE


THE CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

EDITED BY W. THOMSON, M.A.

FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE,
AND PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW.

VOL. V.

(BEING VOL. IX. OF THE CAMBRIDGE MATHEMATICAL JOURNAL.)

Δυοῖν ὀνομάτων μορφὴ μία.

CAMBRIDGE:
MACMILLAN AND Co.;
GEORGE BELL, LONDON;
HODGES AND SMITH, DUBLIN.
1850

mid of XIX century

$n = 3$

Z = 6 general pts.

$\mathcal{P}$: $\pi_P(Z) \subseteq$ conic

CHASLES $\quad \cancel{P \in}$ twisted cubic generated by Z
$\qquad\qquad\qquad\qquad\qquad\qquad \cap$
WEDDLE $\quad P \in$ quartic surface

•

What to do with — Lefschetz properties ?
— "Modern" math. or geom. ?

MACAULAY DUALITY

$Z = \{P_1 \cdots P_6\}$ $\quad$ P
$\downarrow \quad \downarrow \quad\quad \downarrow$
$e_1 \quad e_6 \quad\quad e$

so need $\dim(I_P^2 \cap I_{P_1} \cap \cdots \cap I_{P_6})_2$

R = poly ring

$$\dim\left[R \big/ I_{P_1} \cap \cdots \cap I_{P_6} \cap I_P^2\right]_2 =$$

$$\dim\left[R^\vee \big/ (e_1^2, \ldots, e_6^2, e)\right]_2$$

$$\left[R^\vee \big/ (e_1^2, \ldots, e_6^2)\right]_1 \xrightarrow{e} \left[R^\vee \big/ (e_1^2 \cdots e_6^2)\right]_2 \to$$

$$\to \left[R^\vee \big/ (e_1^2, \ldots, e_6^2, e)\right]_2 \to 0$$

ANALYSIS

$\mathcal{L}_2$ = linear system of quadrics through Z

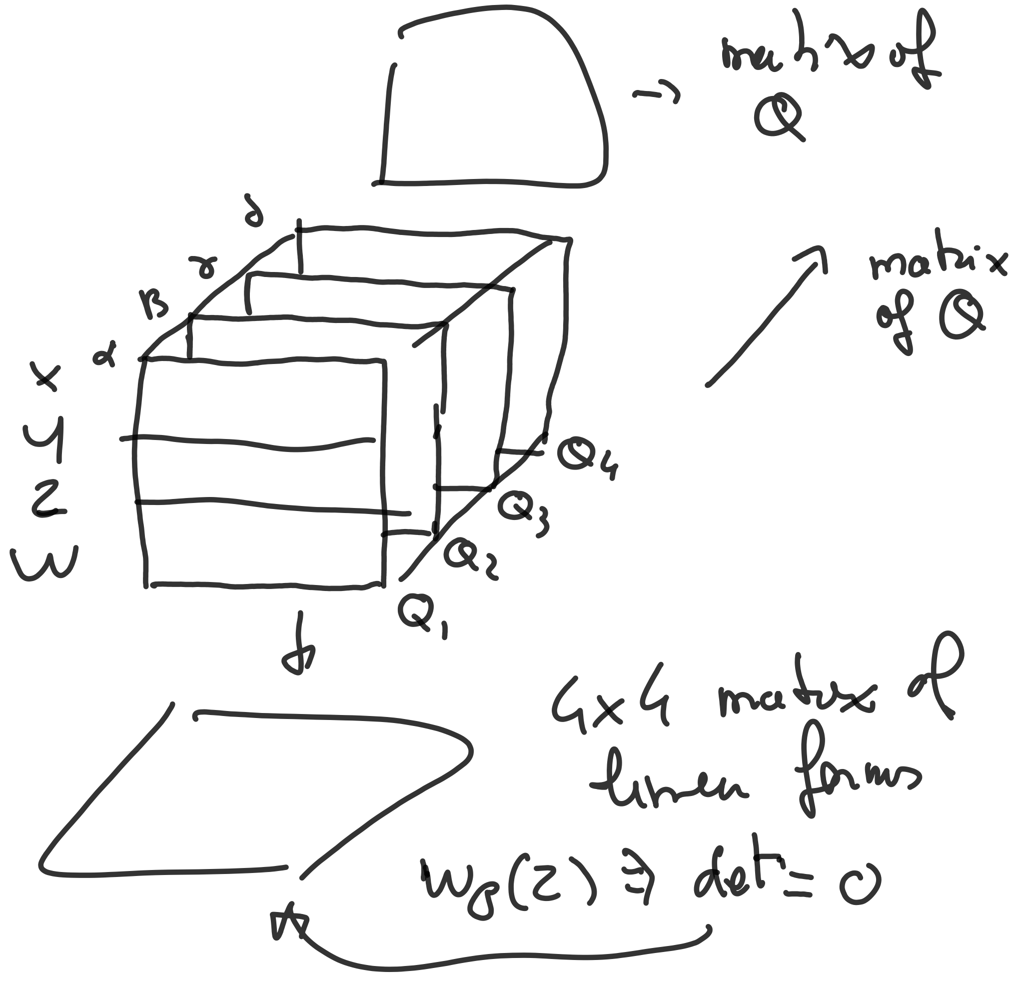
ASSUME

Z imposes indep. conditions to quadrics, so $\dim \mathcal{L}_2 = 4$

- $Q_1, \dots, Q_4$ basis

$\alpha Q_1 + \beta Q_2 + \gamma Q_3 + \delta Q_4$ = general quadric in $\mathcal{L}_2$

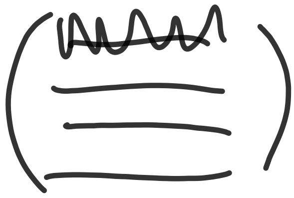
π_P projects Z to a conic $\Leftrightarrow$ $\exists\, Q \in \mathcal{L}_2$ singular at P



$\Rightarrow$ $W_\sigma(Z)$ is defined by the determinant of a 4×4 matrix of linear forms

Questions

Is any quartic a determinant?



→ aCM curve $\deg = 6$, $g = 3$

Is any quartic which is a determinant a Weddle locus? NO

Weddle loci contain lines

Thm (LOPEZ)

OPEN QUESTION

- Characterise Weddle loci in terms of the Noether-Lefschetz locus

GENERALISATIONS

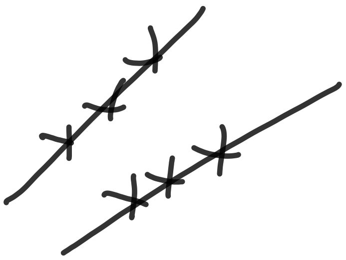
What if Z has
- less than 6 pts? $W_\beta(Z) = \mathbb{P}^3$
- more than 6 pts?

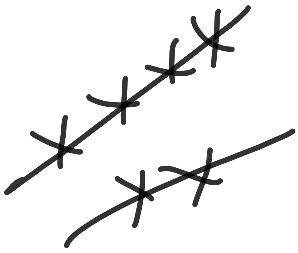
$Z = 7$ pts $\Rightarrow$ expected curve
$Z = 8$ " " finite

---

back to $Z = 6$ pts

What if $\det = 0$ as a polynomial?
$\Downarrow$
(Z non-degenerate) Any proj. sends Z into a conic.





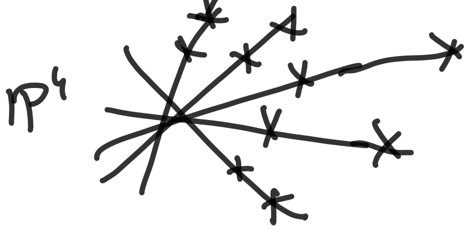
Thm The only examples

Higher dimensions

E.g. $Z = 10$ pts in $\mathbb{P}^4$ and
$W(Z) = \{ P \in \mathbb{P}^4 \quad \pi_P(Z) \subseteq \text{quadric} \}$

Ex $Z = 10$ pts in a rational quartic in $\mathbb{P}^4$

Thm



non-trivial
unique example

Further generalization

What if $\mathcal{L}$ is just *any* linear system (of dim. 4) of quadrics in $\mathbb{P}^3$?

$$\text{Weddle locus of } \mathcal{L} = \left\{ P \in \mathbb{P}^3 : P \text{ is singular for some } Q \in \mathcal{L} \right\}$$

(which can be generalized further to a linear system of quadrics in $\mathbb{P}^n$ of dim. $n+1$)

Thm $\operatorname{rk} < 6 \Rightarrow W$ has at least 4 sing. pts

Open question: Is any determinantal hypersurface of degree $n+1$ the Weddle locus of a suitable linear system of quadrics?

$n=3$
Yes

Open question: Is there a natural interpretation of the Weddle locus of general linear systems as the non-Lefschetz locus of *general* quotient algebras?

Open question

Weddle locus
vs.
rank

What happens if one starts with a linear system of cubics (or quartics, or ...)?

Z 10 pts in $\mathbb{P}^3$

P : proj of $Z \subseteq$ cubic

$\longrightarrow \left(\quad \right)^{16}_{10}$

det $\Longleftrightarrow$ Weddle

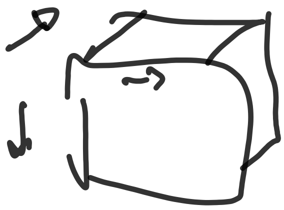
Matrices $\qquad \mathbb{C}^{n,m} = \mathrm{Sym} \oplus \mathrm{Antisym}$

3-dim matrices $\qquad \mathbb{C}^{n,m,p} = \mathrm{Sym} \oplus \mathrm{Antisym} \oplus V \Rightarrow$?

$V = V_1 \oplus V_2$

$\downarrow \qquad \downarrow$

spaces of linear system of quadrics



Basis for $V^{\otimes 3} = e_i \otimes e_j \otimes e_k$.

$$\begin{array}{ccl} Sym^3(V) & \rightarrow & e_i \otimes e_j \otimes e_k, \quad i \leq j \leq k. \\ \Lambda^3 V & \rightarrow & e_i \otimes e_j \otimes e_k, \quad i > j > k. \end{array}$$

$$Q = Q_1 \oplus Q_2,$$

$$\begin{array}{ccl} Q_1 & \rightarrow & e_i \otimes e_j \otimes e_k, \quad i \leq j > k, \\ Q_2 & \rightarrow & e_i \otimes e_j \otimes e_k, \quad i < j \geq k. \end{array}$$

- THANK YOU FOR YOUR ATTENTION

o

o

c