Weighted Path Matrices and Almkvist's Conjecture in Codimension Two

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based on joint work with Nancy Abdallah

Workshop on Lefschetz Properties in Algebra, Geometry, Topology and Combinatorics
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The two parameter family of graded rings $A(m,n)$

- $A(m,n) = \dfrac{\mathbb{R}[e_1,\ldots,e_n]}{(e_1(m),\ldots,e_n(m))}$ Artinian complete intersection

$$e_i = e_i(x_1,\ldots,x_n) = \sum_{1\le k_1<\cdots<k_i\le n} x_{k_1}\cdots x_{k_i}$$

$$e_i(m) = e_i(x_1^m,\ldots,x_n^m)$$

- $H(m,n) = (h_0,\ldots,h_d), \quad h_i = h_i(m,n) = \dim_{\mathbb{R}}(A(m,n)_i)$

$$H(m,n;t) = \sum_{i\ge 0} h_i(m,n)\cdot t^i = \frac{\prod_{i=1}^n(1-t^{mi})}{\prod_{i=1}^n(1-t^i)}$$

$$= \prod_{i=1}^n \left(1+t^i+t^{2i}+\cdots+t^{(m-1)i}\right)$$

$$\Rightarrow h_i(m,n) = \#\text{ partitions of } i \text{ with } \begin{cases}\text{part size} \le n\\ \text{repeated parts} \le m-1\end{cases}$$

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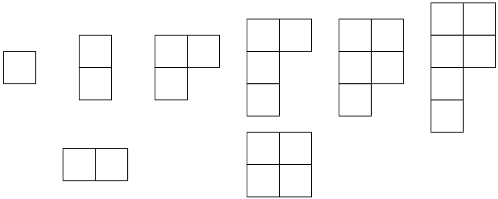
$\mathcal{P}(3,2) = \{ \emptyset$  $\}$

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Almkvist's Conjecture

Conjecture (Almkvist 1980s)

- *For every fixed $m > 0$, $H(m, n)$ is unimodal for all $n \geq 11$.*
- *If m is even, then $H(m, n)$ is unimodal for all n.*

- verified by Almkvist for $3 \leq m \leq 20$ using analysis!
- for $m = 2$ proved independently by Odlyzko-Richmond (analysis), Stanley (SLP), and Hughes(root systems).

Conjecture (Strong Almkvist Conjecture, MCIM 2019)

- *For fixed $m > 0$, $A(m, n)$ has the SLP for all $n \geq 11$.*
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- *$A(m, n)$ has the Lefschetz property w.r.t. the Hilbert function, i.e. "the Hilbert Lefschetz property" (HLP)*

$$\operatorname{rank}\left(\times \ell^k : A_i \to A_{i+k}\right) = \min\{h_i, \ldots, h_{i+k}\}, \ \forall i, k$$

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Connection to Complex Reflection Groups

- group of $n \times n$ "m^{th}-root of unity" permutation matrices

$$G(m,1,n) = \left\{ \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \middle| \lambda_i^m = 1 \right\} \rtimes \mathfrak{S}_n$$

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- *The group $G(2,1,n)$ is the Weyl group of type B_n.*
- *The ring $A(2,n) \cong H^{2\bullet}(G/P,\mathbb{R})$ the cohomology ring of a Kähler manifold.*

Corollary

$A(2,n)$ satisfies SLP (and HRR) for all n.

$\Rightarrow H(2,n;t) = (1+t)(1+t^2)\cdots(1+t^n)$ *is unimodal* $\forall n$

(no elementary proof known)

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Is $A(m,n)$ a cohomology ring of a Kähler manifold for $m > 2$?

Answer

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- For $m = 2m'$ even:
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$$A(m,2) = \frac{\mathbb{R}[e_1,e_2]}{(e_1(m),e_2(m))} = \frac{\mathbb{R}[e_1,e_2]}{(f_m(e_1,e_2),e_2^m)} = \frac{\mathbb{R}[e_1,e_2]}{\mathrm{Ann}(F_m(E_1,E_2))}$$

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$$f_m(e_1, e_2) = e_1(m) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \frac{m}{m-k} \binom{m-k}{k} e_1^{m-2k} e_2^k$$

Lemma (Abdallah-M.)

$$F_m(E_1, E_2) = \sum_{n=0}^{m-1} \frac{m}{m+2n} \binom{m+2n}{n} \frac{E_1^{m+2n-1}}{(m+2n-1)!} \cdot \frac{E_2^{m-n-1}}{(m-n-1)!}$$

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We asked MathStackExchange how to prove the identities

$$\sum_{k=0}^{i}(-1)^k \frac{m}{m-k}\binom{m-k}{k}\frac{m}{m+2(i-k)}\binom{m+2(i-k)}{i-k} = 0, \ \forall 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor.$$

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$$C(x) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^n$$

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- $\ell = e_1 \in A(m,2)$ is the only linear form (up to scaling)!
- for $d = 3(m-1)$ and $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$ it has the monomial basis

$$\mathcal{B}_i = \left\{ b_p^i = e_1^{i-2p} e_2^p \mid \left\lfloor \frac{i+2-m}{2} \right\rfloor_\star \leq p \leq \left\lfloor \frac{i}{2} \right\rfloor \right\}.$$

- and the i^{th} Hessian matrix

$$\mathrm{Hess}_i(F_m, \mathcal{B}_i) = \left(b_p^i b_q^i \circ F_m \right)_{p,q}$$

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- $G = (\mathcal{V}, \mathcal{E})$ directed, acyclic graph
- $\omega \colon \mathcal{E} \to R$ edge weighting (R commutative ring)
- Row and column vertex sets $\mathcal{V} \supseteq \begin{cases} \mathcal{A} = \{A_1, \ldots, A_r\} \\ \mathcal{B} = \{B_1, \ldots, B_r\} \end{cases}$
- directed path $P \colon A_i \to B_j \Rightarrow \omega(P) := \prod_{e \in P} \omega(e)$
- weighted path matrix $= r \times r$ matrix

$$W = W(G, \omega, \mathcal{A}, \mathcal{B}) = \left(\sum_{P \colon A_p \to B_q} \omega(P) \right)_{1 \le p,q \le r}$$

- path system $\mathcal{P} = \left\{ P_i \colon A_i \to B_{\sigma(i)} \right\}_{i=1}^{r}$
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Subdiagonal NE Lattice Paths

- $\mathcal{V} = \{(a, b) \in \mathbb{Z}^2 \mid a \geq b\}$, $\mathcal{E} =$ unit steps up or to the right.
- directed paths = subdiagonal NE lattice paths
- $\omega \equiv 1$

$$\mathcal{A}^i_m = \left\{ A_p = (p, p) \;\middle|\; \left\lfloor \frac{i+2-m}{2} \right\rfloor_* \leq p \leq \left\lfloor \frac{i}{2} \right\rfloor \right\}$$

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$$= (d-2i)! \cdot \mathrm{Hess}_i(F_m, \mathcal{B}_i)|_{(1,0)}$$

Subdiagonal NE Lattice Paths

- $\mathcal{V} = \{(a,b) \in \mathbb{Z}^2 \mid a \geq b\}$, $\mathcal{E} =$ unit steps up or to the right.
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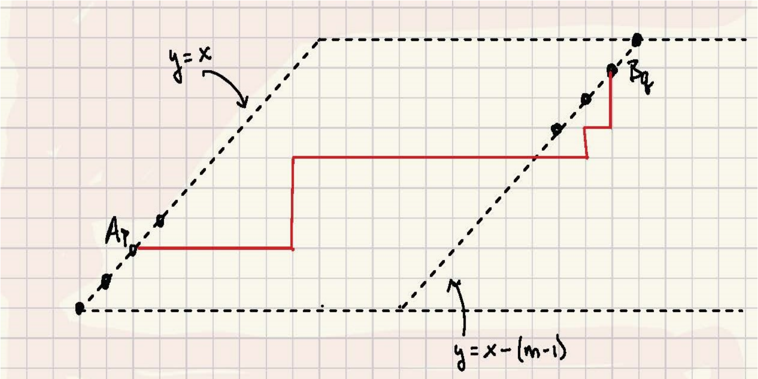

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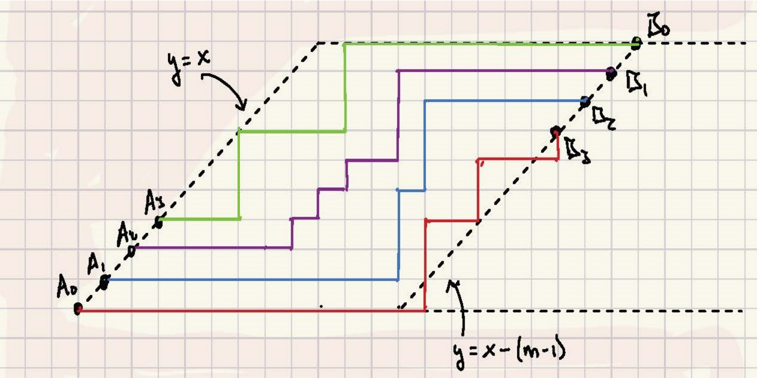

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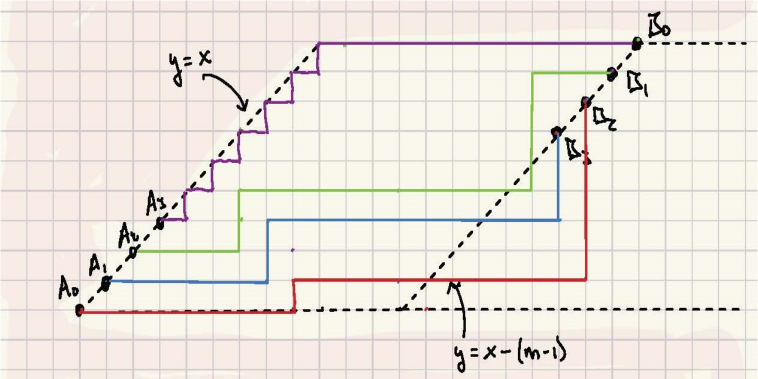

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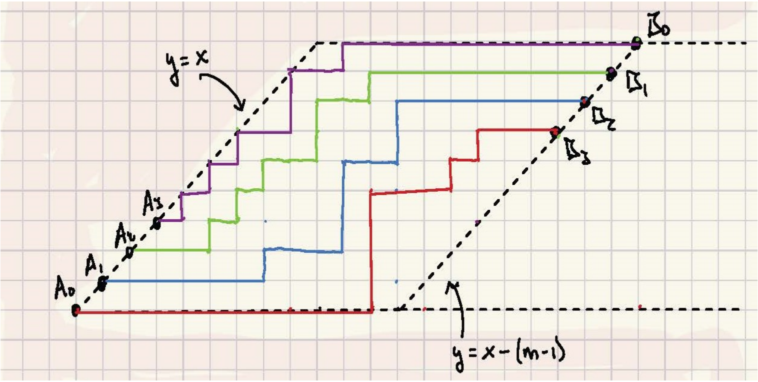

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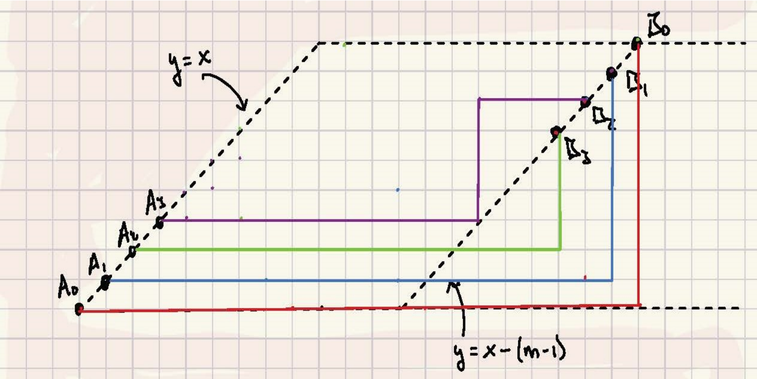

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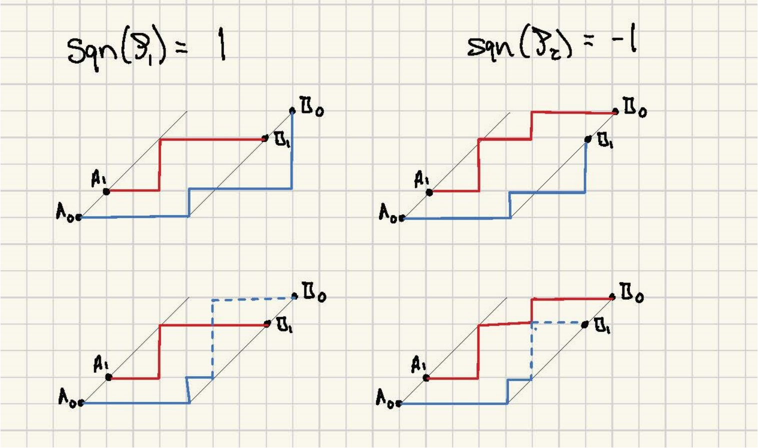

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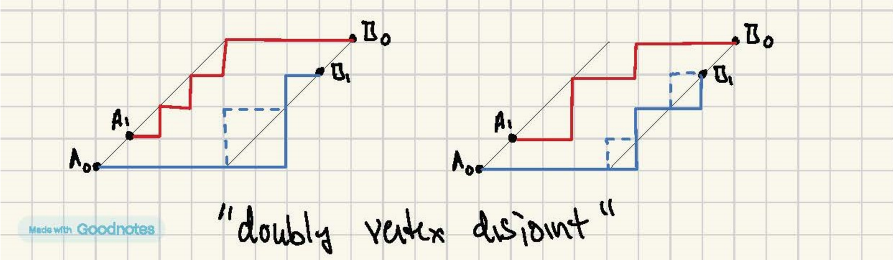

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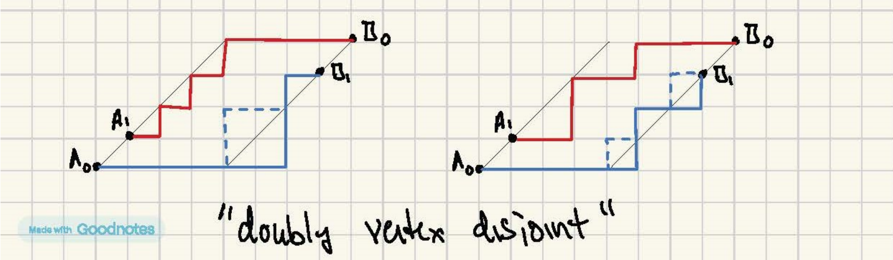

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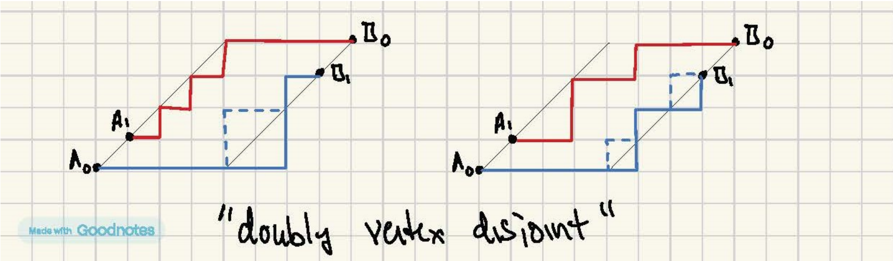

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Hessian Determinant

Theorem (Abdallah-M.)

For every $0 \leq i \leq \left\lfloor \frac{3(m-1)}{2} \right\rfloor$,

$$\det\left(\mathrm{Hess}_i(F_m, \mathcal{B}_i)\right)|_{(1,0)} = \left(\frac{1}{(d-2i)!}\right)^{h_i} \cdot (-1)^{\left\lfloor \frac{h_i}{2} \right\rfloor} \cdot N(i,m)$$

where $h_i = \left\lfloor \frac{i+2}{2} \right\rfloor - \left\lfloor \frac{i+2-m}{2} \right\rfloor_* = H(m,2)_i = \#\left(\mathcal{A}_m^i\right)$ *and*

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$\times e_1^{d-2i} : A(m,2)_i \to A(m,2)_{d-i}$ isomorphism $\Leftrightarrow$ $N(i,m) \neq 0$

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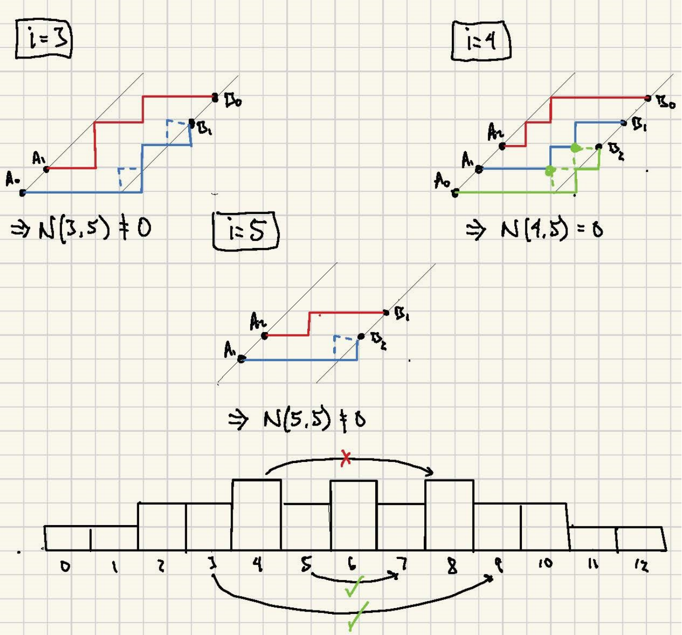
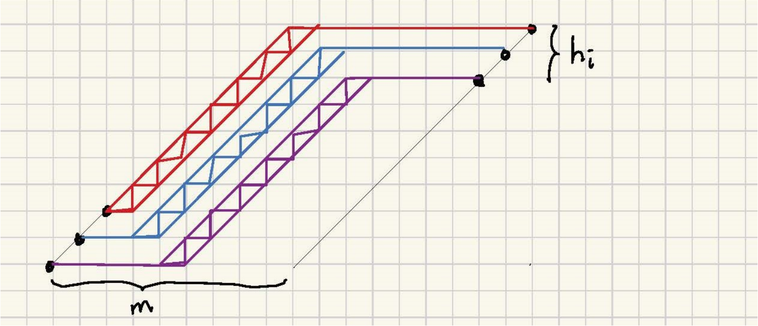


Existence of Doubly Vertex Disjoint Path Systems



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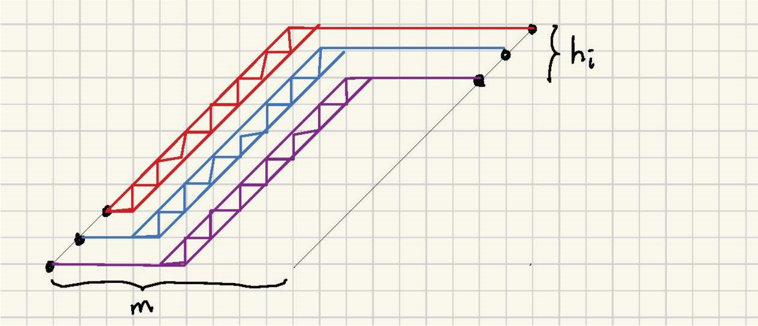

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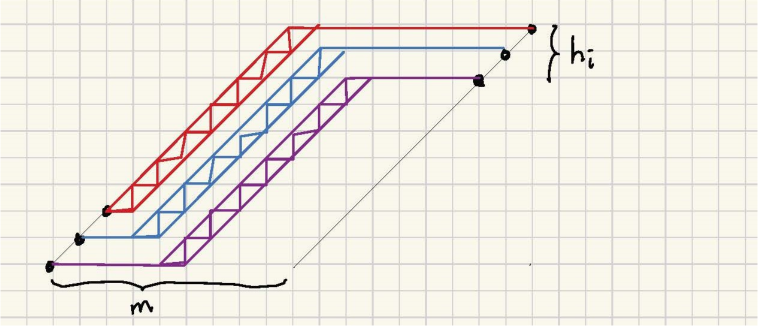

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- $A = \bigoplus_{i=0}^{d} A_i$, $\int_A : A_d \to \mathbb{R}$ graded AG algebra $/\mathbb{R}$
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- i^{th} primitive subspace $P_{i,\ell} = \ker\left(\times \ell^{d-2i+1} : A_i \to A_{d-i+1}\right)$
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Cohomology Rings and Hodge-Riemann Relations

Fact

If $A = H^{2\bullet}(X, \mathbb{R})$, the cohomology ring of a Kähler manifold then A satisfies SLP and HRR.

Theorem (Abdallah-M.)

If $m > 2$ then $A(m, 2)$ does NOT satisfy HRR.

Proof.

- $m > 2 \Rightarrow 0 \leq i = 2 \leq \left\lfloor \frac{3(m-1)}{2} \right\rfloor$ and $\begin{cases} h_0 = h_1 = 1 \\ h_2 = 2 \end{cases}$
- our Theorem $\Rightarrow \operatorname{sgn}(\det(\mathrm{Hess}_2(F_m, \mathcal{B}_2))) = (-1)^{\left\lfloor \frac{h_2}{2} \right\rfloor} = -1$
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Complex Hodge-Riemann Relations for $A(m, 2)$

Theorem (Abdallah-M.)

If m is even then $A(m, 2)$ satisfies the complex HRR, meaning for all $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$

$$\operatorname{sgn}\left(\operatorname{Hess}_i(F_m, \mathcal{B}_i)\right) = \sum_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor} (-1)^j (h_{2j} - h_{2j-1})$$

$$\Leftrightarrow \ (\sqrt{-1})^i \cdot (\alpha, \alpha)_i^\ell > 0, \ \forall 0 \neq \alpha \in P_{i,\ell}.$$

Open Problems and Questions

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- How are doubly vertex disjoint lattice paths related to partitions?
- Find Macaulay dual generator of $A(m, n)$ for $n > 2$.
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