ARTINIAN COX-GORENSTEIN ALGEBRAS

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G-graded algebras

Let $(G, +)$ be an Abelian group, possibly with torsion.

Definition. Let A be a $\Bbbk$-algebra, a G-grading in A is a decomposition of A into $\Bbbk$-vector spaces parameterized by G, i.e.,

$$A = \bigoplus_{g \in G} A_g$$

such that the product in A satisfies $A_g A_h \subseteq A_{g+h}$. We always consider $A_0 = \Bbbk$ and any element in A_g for some $g \in G$ we call it homogeneous of degree g.

Example. Set $A = \Bbbk[x, y]$ with $\deg(x) = (1, 0)$ and $\deg(y) = (0, 1)$. These induce a $\mathbb{Z}^2$-grading on A.

Cox rings, Toric varieties and Cox algebras

Let $\mathbb{k}$ be an algebraic closed field with characteristic zero.

Definition. Let Z be a complete normal variety with finitely generated Class group. The Cox ring of Z is the $\mathrm{Cl}(Z)$-graded ring

$$\mathrm{Cox}(Z) := \bigoplus_{[D]\in \mathrm{Cl}(Z)} H^0(\mathcal{O}_Z(D)).$$

Theorem[Berchtold and Hausen 2003]. Let Z be a normal variety A_2-maximal (e.g., projective) with $\mathcal{O}_Z(Z) = \mathbb{k}$ and finitely generated class group. Then Z is a toric variety if and only if $\mathrm{Cox}(Z)$ is a polynomial ring.

Definition. A G-graded algebra A isomorphic to the quotient of the Cox ring of a toric variety (polynomial ring) by a homogeneous ideal is a Cox Algebra.

Hilbert function and Hasse-diagram

We know that any finitely generated Artinian (G-graded) $\mathbb{k}$-algebra A has finite dimension over $\mathbb{k}$, therefore it is finitely graded and every graded piece is a finite-dimensional $\mathbb{k}$-vector space and we set $h_g = \dim A_g$. The Hilbert function of A:

$$\mathrm{HF}_A : G \to \mathbb{Z}_+$$

is defined by $\mathrm{HF}_A(g) = h_g$. Now, let $(G, +, \preceq)$ be a partially ordered Abelian group. In order to properly encode the structural information contained in the Hilbert function we introduce the Hasse-Hilbert diagram of A defined as a vertex-weighted directed graph structure over the covering graph of G where the weight of a vertex g is the Hilbert function h_g.

Hasse-Hilbert diagram

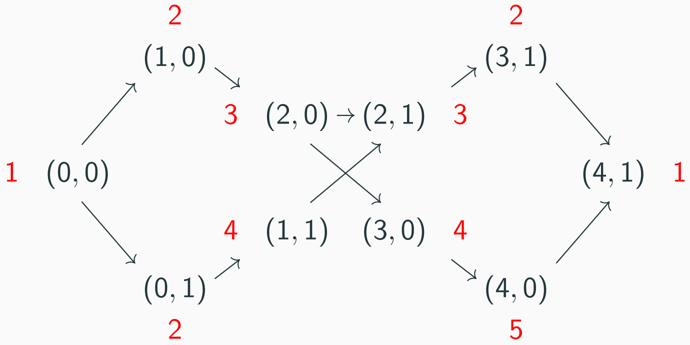
By definition of the covering graph of a partial order, the vertex set is the POSET, in our case all $g \in G$ such that $A_g \neq 0$, and two vertices $g, h \in G$ are connected if they are immediate neighbors, that is, they are comparable and there is no other comparable element between them. As usual, a maximal element in G is an element $g \in G$ for which $h \in G$ such that $g \preceq h$ does not exist, with $g \neq h$. We say that a maximal element $g \in G$ is the greatest element in G if $h \preceq g$ for all $g \in G$.

Remark. We are interested in the case when G is equal to the class group $\mathrm{Cl}(Z)$ of a toric variety Z, in this case we have a natural partial order:

For $\alpha, \beta \in \mathrm{Cl}(Z)$, $\alpha \preceq \beta$ if and only if $\beta - \alpha$ is an effective divisor.

Example

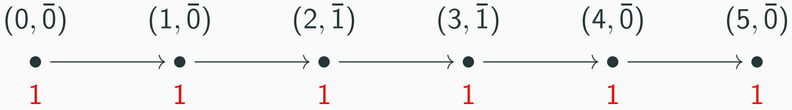
Let $S = \Bbbk[x,y,u,v]$ be $\mathbb{Z}^2$-graded by $\deg(x) = \deg(y) = (1,0)$ and $\deg(u) = \deg(v) = (0,1)$. We consider $G = \mathbb{Z}^2$ with the partial order $(a,b) \preceq (c,d)$ if and only if $a \leq c$ and $b \leq d$. Let $I = (S_{(0,2)}, S_{(5,0)}, x^2u - y^2v, x^2v, y^2u) \subset S$, the quotient $A = S/I$ is an Artinian $\mathbb{Z}^2$-graded algebra. The Hasse-Hilbert diagram has $A_{(4,1)}$ as the greatest element. In fact $A = \bigoplus_{g \leq (4,1)} A_g$.



Example with torsion

Let $S = \mathbb{k}[x, y, z]$ be the polynomial ring with a $G = \mathbb{Z} \oplus \mathbb{Z}_2$-grading given by $\deg(x) = (1, \overline{1})$, $\deg(y) = (1, \overline{0})$ and $\deg(z) = (2, \overline{1})$. We consider G with the order $(a, \overline{b}) \preceq (c, \overline{d})$ if and only if $a \leq c$. Let $I = (x, y^2, z^3) \subset S$ and $A = S/I$. It is easy to see that A is an Artinian G-graded algebra. The Hasse-Hilbert diagram of A is linear and we can write

$$A = A_{(0,\overline{0})} \oplus A_{(1,\overline{0})} \oplus A_{(2,\overline{1})} \oplus A_{(3,\overline{1})} \oplus A_{(4,\overline{0})} \oplus A_{(5,\overline{0})}.$$



Artinian Cox-Gorenstein Algebras

Definition. Let $A = \mathbb{k}[X_1, \ldots, X_n]/I = \oplus_{g \in G} A_g$ be an Artinian G-graded $\mathbb{k}$-algebra and let $\mathfrak{m} := (\overline{X_1}, \ldots, \overline{X_n}) \subset A$. We say that A is Cox-Gorenstein if there exists $\omega \in G$ such that $\mathrm{soc}(A) := (0 : \mathfrak{m}) = A_\omega \simeq \mathbb{k}$. In this case, ω is called the socle degree of A and I a Cox-Gorenstein ideal.

Theorem. Let $A = \oplus_{g \in G} A_g$ be an Artinian G-graded $\mathbb{k}$-algebra. A is Cox-Gorenstein if and only if A has the Poincaré duality.

Theorem. Let Z be a d-dimensional projective toric orbifold, and assume that $f_i \in S_{\beta_i} = H^0(\mathcal{O}_Z(\beta_i))$ for $i = 0, ..., d$ where $\beta_i \in \mathrm{Cl}(Z)$ is an ample class and the f_i don't vanish simultaneously on Z then

$$(S/(f_0, \ldots, f_d))_\omega \simeq \mathbb{C}$$

for $\omega = \sum_{i=0}^{d} \beta_i - \beta_0$ where β_0 is the anticanonical class of Z. Moreover, for each variable x_i ,

$$x_i \cdot S_\omega \in (f_0, \ldots f_d).$$

Toric Macaulay duality theorem

Corollary. If Z has Picard rank 1 then $A = S/(f_0, \ldots, f_d)$ is an Artinian Cox-Gorenstein algebra with socle degree ω.

Toric varieties with Picard rank one are weighted projective spaces and fake projective spaces.

The assumption on the Picard rank is essential.

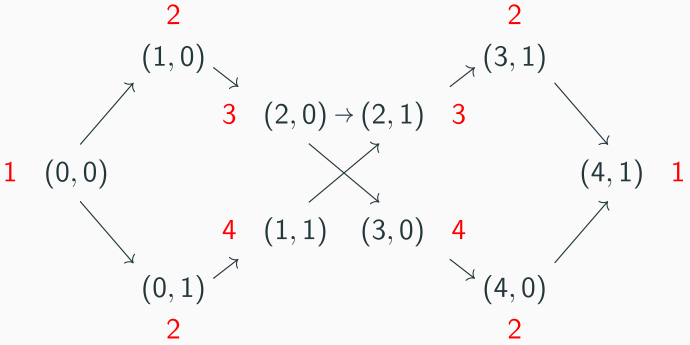
Example. For $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates (x, y, u, v) we consider $(f_0, f_1, f_2) = (x^2u - y^2v, x^2v, y^2u)$ then $A = S/(x^2u - y^2v, x^2v, y^2u)$ is non-Artinian.

Corollary. If Z has Picard rank bigger than 1 there exists a minimal Artinian Cox-Gorenstein ideal with socle degree ω containing $(f_0, \ldots, f_d)$.

We mean minimal in the sense that we can consider a natural Artinian Cox-Gorenstein reduction, preserving the configuration of the Hasse-Hilbert diagram and preserving the pairings which were already perfect.

Example: Minimal Artinian Cox-Gorenstein reduction

The Artinian minimal reduction of $I = (x^2u - y^2v, x^2v, y^2u)$ is $I_{\mathcal{A}} = (S_{(0,2)}, S_{(5,0)}, x^2u - y^2v, x^2v, y^2u)$ and its Gorenstein minimal reduction is $I_{\mathcal{G}} = (I_{\mathcal{A}}, x^4, x^2y^2, y^4)$ and its Hasse-Hilbert diagram is:



A Lefschetz type theorem for toric varieties

Theorem. [Batyrev and Cox 1994] Let X be a quasi-smooth ample hypersurface in a d-dimensional projective toric orbifold Z with X cut off by $f \in S_\beta$, then the natural map $i^* : H^i(Z) \to H^i(X)$ is an isomorphism for $i < d-1$ and an injection for $i = d-1$. Moreover i^* is a morphism of Hodge structures.

Definition. The primitive cohomology group $H^{d-1}_{prim}(X)$ is defined by the exact sequence

$$0 \to H^{d-1}(Z) \to H^{d-1}(X) \to H^{d-1}_{\mathrm{prim}}(X) \to 0.$$

We denote by $H^{p,d-1-p}_{\mathrm{prim}}(X, \mathbb{Q})$ the Hodge components of the primitive cohomology with rational coefficients. Then by the Noether-Lefschetz theorem for toric varieties on a very general hypersurface in a $(2k+1)$-dimensional projective toric orbifold with degree β such that $\beta - \beta_0$ is nef, we have that $H^{k,k}_{\mathrm{prim}}(X, \mathbb{Q}) = 0$.

Jacobian ring and primitive cohomology

Theorem. [Batyrev and Cox 1994] Let Z be a d-dimensional projective toric orbifold, and let $X \subset Z$ be a quasi-smooth ample hypersurface cut off by $f \in S_\beta$. If $R(f)$ is the Jacobian ring of f, then for $p \neq d/2 - 1$, there is a canonical isomorphism

$$R(f)_{(d-p)\beta-\beta_0} \cong H^{p,d-1-p}_{\mathrm{prim}}(X).$$

Proposition. There exists a $\mathrm{Cl}(Z)$-graded Artinian Cox-Gorenstein ideal E containing $R(f)$.

This E appears naturally in the study of the deformation of Hodge classes, i.e., deformation of cohomology classes of type (k,k) for some $k \in \mathbb{N}$. For example on $\mathbb{P}^n$ in Noether-Lefschetz Locus and a Special Case of the Variational Hodge Conjecture: Using Elementary Techniques by A. Dan.

Tangent space of a point in the Noether-Lefschetz locus

Let Z be a $(2k+1)$-dimensional projective toric orbifold.

Definition. [Noether-Lefschetz locus]
$NL_\beta = \{$X a quasi-smooth hypersurface with deg(X)=$\beta \mid H^{k,k}_{\mathrm{prim}}(X,\mathbb{Q}) \neq 0\}$.

Example. For $Z = \mathbb{P}^3$ and $\beta - 4 \geq 0$, i.e., the classical Noether-Lefschetz locus, NL_β is equal to the locus of smooth surfaces with degree $\beta \in \mathrm{Pic}(\mathbb{P}^3) \simeq \mathbb{Z}$ and Picard rank strictly bigger than 1.

Theorem. $T_{[f]}(NL_\beta) \cong E_\beta$, where

$$E = \{K \in S_\bullet \mid \sum_{i=1}^{h_{2k}(X,\mathbb{Q})} \lambda_i \int_{\mathrm{Tub}\,\gamma_i} \frac{KR\Omega_0}{f^{k+1}} = 0 \textit{ for all } R \in S_{\omega-\bullet}\},$$

$\omega = (k+1)\beta - \beta_0$ and Tub$(-)$ is the adjoint of the residue map.

Toric Macaulay-Matlis theorem

Let G be an Abelian group and let $S = \mathbb{k}[x_1, \ldots, x_s]$ be the polynomial ring with a G-grading. Let $Q = \mathbb{k}[X_1, \ldots, X_s]$ be the ring of differential operators associated to S, that is, $X_i = \frac{\partial}{\partial x_i}$, and S has a natural structure of Q-module given by differentiation $X_i(x_j) = \delta_{ij}$. The G-grading on S induces a G-grading on Q by defining $\deg(X_i) = \deg(x_i) \in G$.

Theorem. We have a bijective correspondence

$$\begin{array}{rcl} \{G\text{-homogeneous ideals } I \subset Q\} & \leftrightarrow & \{G\text{-graded Q-submodules of } S\} \\ I & \mapsto & I^{-1} := \{f \in S \mid \alpha(f) = 0 \ \forall \alpha \in I\} \\ \{\alpha \in Q \mid \alpha(f) = 0 \ \forall f \in M\} =: Ann(M) & \leftarrow\!\shortmid & M \end{array}$$

Under this correspondence, $M = I^{-1}$ is finitely generated as an S-module if and only if $A = Q/I$ is Artinian. Moreover, A is Artinian Cox-Gorenstein if and only if $M = Q \cdot f$ is a cyclic module.

Lefschetz Properties: Preliminaries

Let $A = Q/I = \oplus_{g \in G} A_g$ be a Cox algebra and let $\mathcal{L} = \langle X_1, \ldots, X_s \rangle \subseteq A$ be the $\mathbb{k}$-linear subspace generated by the class of the variables in Q. We say that any homogeneous element $L \in \mathcal{L}$ is linear.

Also we say that two graded pieces of A, A_g and A_h, are :

- linearly consecutive if $g \preceq h$ and $\mathcal{L}_{h-g} \neq 0$;
- linearly comparable if $g \preceq h$ and there is $L \in \mathcal{L}_l$ such that $h = g + kl$ for some $k \in \mathbb{Z}_+$.

Example. Consider $Q = \mathbb{k}[X, Y, U, V]$ and $G = \mathbb{Z}^2$ with the partial order $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \leq d$, and the G-grading given by $\deg(X) = \deg(Y) = (1, 0)$ and $\deg(U) = \deg(V) = (0, 1)$. Then $(0, 0)$ is linearly comparable with $(1, 1)$ but not linearly consecutive.

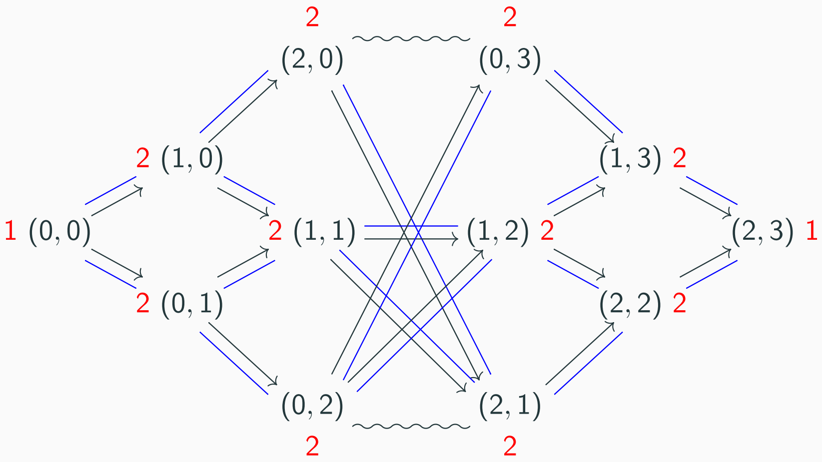
Remark. The $\mathbb{k}$-linear subspace $\mathcal{L}$ has a natural disjoint decomposition.

Toric Weak Lefschetz Property

Definition. Let A be an Artinian Cox algebra. We say that A has the Toric Weak Lefschetz property (TWLP) if for every linearly consecutive summands of A, A_g and A_h, there is a linear element $L \in \mathcal{L}_{h-g}$ such that the $\mathbb{k}$-linear multiplication map $\bullet L : A_g \to A_h$ has maximal rank.

Example. Consider $S = \mathbb{k}[x, y, u, v]$ and $G = \mathbb{Z}^2$ and a G-grading given by $\deg(x) = \deg(y) = (1, 0)$ and $\deg(u) = \deg(v) = (0, 1)$. Let $f \in S_{(2,3)}$ be given by $f = x^2u^3 + y^2v^3$. Let $Q = \mathbb{k}[X, Y, U, V]$ be the ring of differential operators acting on S and let $I = \mathrm{Ann}(f) \in Q$ be the Artinian Cox-Gorenstein ideal producing $A = Q/I$ and the generators of the linear elements $\mathcal{L}_{(1,0)} = \langle X, Y\rangle$ and $\mathcal{L}_{(0,1)} = \langle U, V\rangle$. We have that $L_{(0,1)} = U + V$ or $L_{(1,0)} = X + Y$ are linear elements. Moreover, for every linearly consecutive degrees h and g with $g \preceq h$, $h - g = (0, 1)$ or $h - g = (1, 0)$ and the multiplication maps $\bullet L_{(0,1)}$ or $\bullet L_{(1,0)}$ have maximal rank.

Example: Toric Weak Lefschetz Property



Hasse-Hilbert diagram and linear comparability

Toric Strong Lefschetz Property

Definition. Let A be an Artinian Cox algebra. We say that A has the Toric Strong Lefschetz property (TSLP) if for every linearly comparable summands A_g and A_h of A, there is a linear element $L \in \mathcal{L}_l$ with $h = g + kl$ such that the $\Bbbk$-linear multiplication map $\bullet L^k : A_g \to A_h$ has maximal rank.

Example. Let $S = \Bbbk[x, y, z]$ be $\mathbb{Z}$-graded by $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$. Let $f \in S_4$ given by $f = x^4 + y^4 + z^2$. In the dual $Q = \Bbbk[X, Y, Z]$ we obtain

$$\mathrm{Ann}(f) = (XY, XZ, YZ, X^5, Y^5, Z^3, X^4 - Y^4, X^4 - Z^2).$$

Let $A = Q/I$ be the Cox-Gorenstein algebra associated with f. We have $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4$ with $A_1 = \langle X, Y\rangle$, $A_2 = \langle X^2, Y^2, Z\rangle$, $A_3 = \langle X^3, Y^3\rangle$, and $A_4 = \langle X^4\rangle \simeq \langle Y^4\rangle \simeq \langle Z^2\rangle$. It is easy to verify that A has the TSLP with the linear element $L = X + Y$.

Toric Hessian Criterion

Definition. Let $\mathcal{B} = \{\beta_1, \dots, \beta_s\}$ and $\mathcal{C} = \{\gamma_1, \dots, \gamma_t\}$ be $\Bbbk$-basis of A_g and $A_{g'}$ respectively. The toric mixed Hessian of $f \in S$ with mixed order (β, γ) is $Hess_f^{(\mathcal{B},\mathcal{C})} := [\beta_i \circ \gamma_j(f)]$.
A linear element L is ϕ-linear if there exists $\phi \in \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q})$ such that $\phi(\deg(L)) = 1$.

Theorem. Let $A = Q/I$ with $I = \mathrm{Ann}(f)$ be an Artinian Cox-Gorenstein G-graded $\Bbbk$-algebra. Let A_g and A_h be two linearly comparable graded pieces of A such that $h = g + kl$, and let $L = a_1X_1 + \ldots + a_mX_m \in \mathcal{L}_l$ be a ϕ-linear element such that $\phi(\deg(f)) \in \mathbb{Z}_+$. Then the matrix of the $\Bbbk$-linear map $\bullet L^k : A_g \to A_h$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ can be given by:

$$\left[\bullet L^k\right]_{\mathcal{B}}^{\mathcal{C}} = k! \cdot Hess_f^{(\mathcal{C}^*,\mathcal{B})}(a) \quad \text{where} \quad a = (a_1, \dots, a_m, 0, \dots, 0).$$

Dziękuję

Thank you for the attention!

Some References

- Berchtold, F. and Hausen, J., *Cox rings and combinatorics*, Transactions of the American Mathematical Society, 2003.
- Bruzzo, U. and Montoya, W., *On the Hodge conjecture for quasi-smooth intersections in toric varieties*, São Paulo J. Math. Sci., 2021.
- Cox, D., Little, J. and Shenck H., *Toric Varieties* Graduate studies in mathematics. American Mathematical Society, 2011.
- Dan, A., *Noether-Lefschetz Locus and a Special Case of the Variational Hodge Conjecture: Using Elementary Techniques*, Analytic and Algebraic Geometry, Springer Singapore, 2017.
- Gondim R. and Zappalà G., *On mixed Hessians and the Lefschetz properties* J. Pure Appl. Algebra 223, 10, 2019.
- Harima, T., Maeno, T., Morita, H., Numata, Y., Wachi, A. and Watanabe, J., *The Lefschetz Properties*, Springer Berlin Heidelberg , 2013.